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METHOD OF INTEGRAL ALGEBRAICAL FUNCTIONS IN THE THEORY OF ELASTICITY OF THREE DIMENSIONS

By S. A. Savin

1. **Formulas of Displacements and Unit Stresses.** In the present article we are considering a homogeneous isotropic solid in equilibrium under the influence of external forces and only a single volume force which is its own weight. The own weight per unit volume denoted by q is therefore a constant at all points of such solid, i.e.

$$q = \text{Const.} \tag{1}$$

Some problems of the theory of elasticity, as known, may be solved by means of integral algebraical functions or, in other words, of integral non-homogeneous polynomials. For such kind of problems in the author's article "Abridged Formulas for the Solution of Differential Equations of the Theory of Elasticity by means of Integral Polynomials" (Journal of Mathematics and Physics, vol. XVII, number 4, January 1939) there were given the expressions of displacements u, v, w by means of integral polynomials of degree ω, which may be written in the symbolic form

$$[u, v, w] = \sum_{0}^{\omega} [A_{\alpha\epsilon\nu}, B_{\alpha\epsilon\nu}, C_{\alpha\epsilon\nu}] x^{\alpha} y^{\epsilon} z^{\nu}, \tag{2}$$

where the integral positive indices α, ϵ, ν of the displacement coefficients

$$A_{\alpha\epsilon\nu}, B_{\alpha\epsilon\nu}, C_{\alpha\epsilon\nu}, \tag{3}$$

and their sum $\alpha + \epsilon + \nu$, called as an order of the coefficient, are subject to condition

$$0 \leq [\alpha, \epsilon, \nu, \alpha + \epsilon + \nu] \leq \omega. \tag{4}$$

The expressions of partial derivatives of the first and second orders of an integral algebraical function were given there in an abridged form by the formulas (3) and (5), and the expression of the cubic dilatation Δ at the current point of a solid was given by the formula (18).

Using these formulas we obtain the expressions of the normal unit stresses

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$$\left.\begin{aligned} X_x &= \lambda\cdot\Delta + 2\mu\frac{\partial u}{\partial x} = \sum_0^{\omega-1} H_{\alpha\nu\rho}\cdot x^\alpha y^\nu z^\rho, \\ Y_y &= \lambda\cdot\Delta + 2\mu\frac{\partial v}{\partial y} = \sum_0^{\omega-1} K_{\alpha\nu\rho}\cdot x^\alpha y^\nu z^\rho, \\ Z_z &= \lambda\cdot\Delta + 2\mu\frac{\partial w}{\partial z} = \sum_0^{\omega-1} L_{\alpha\nu\rho}\cdot x^\alpha y^\nu z^\rho, \end{aligned}\right\} \tag{5}$$

the coefficients of which being determined by the formulas

$$\left.\begin{aligned} H_{\alpha\nu\rho} &= \lambda(\nu+1)C_{\alpha(\nu+1)\rho} + (\lambda+2\mu)(\alpha+1)A_{(\alpha+1)\nu\rho} \\ &\quad + \lambda(\epsilon+1)B_{\alpha(\epsilon+1)\rho}, \\ K_{\alpha\nu\rho} &= \lambda(\alpha+1)A_{(\alpha+1)\nu\rho} + (\lambda+2\mu)(\epsilon+1)B_{\alpha(\epsilon+1)\rho} \\ &\quad + \lambda(\nu+1)C_{\alpha(\nu+1)\rho}, \\ L_{\alpha\nu\rho} &= \lambda(\epsilon+1)B_{\alpha(\epsilon+1)\rho} + (\lambda+2\mu)(\nu+1)C_{\alpha(\nu+1)\rho} \\ &\quad + \lambda(\alpha+1)A_{(\alpha+1)\nu\rho}, \end{aligned}\right\} \tag{6}$$

and the expressions of the tangential or shearing unit stresses

$$\left.\begin{aligned} X_y = Y_x &= \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \sum_0^{\omega-1} R_{\alpha\nu\rho}\cdot x^\alpha y^\nu z^\rho, \\ Y_z = Z_y &= \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \sum_0^{\omega-1} P_{\alpha\nu\rho}\cdot x^\alpha y^\nu z^\rho, \\ Z_x = X_z &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \sum_0^{\omega-1} Q_{\alpha\nu\rho}\cdot x^\alpha y^\nu z^\rho, \end{aligned}\right\} \tag{7}$$

the coefficients of which being determined by the formulas

$$\left.\begin{aligned} R_{\alpha\nu\rho} &= \mu[(\alpha+1)B_{(\alpha+1)\nu\rho} + (\epsilon+1)A_{\alpha(\epsilon+1)\rho}], \\ P_{\alpha\nu\rho} &= \mu[(\epsilon+1)C_{\alpha(\epsilon+1)\rho} + (\nu+1)B_{\alpha(\nu+1)\rho}], \\ Q_{\alpha\nu\rho} &= \mu[(\nu+1)A_{\alpha(\nu+1)\rho} + (\alpha+1)C_{(\alpha+1)\nu\rho}]. \end{aligned}\right\} \tag{8}$$

Lamé's coefficients of elasticity are denoted here by λ and μ. By means of Young's modulus E, Poisson's ratio σ, and modulus of elasticity in shear G used in engineering, the coefficients λ and μ are expressed by well known formulas

$$\lambda = \frac{\sigma E}{(1+\sigma)(1-2\sigma)}, \qquad \mu = G = \frac{E}{2(1+\sigma)}. \tag{9}$$

From (5) and (7) it is seen that the formulas (6) and (8) must be applied to all possible combinations of the integral positive indices α, ϵ, ν connected by the condition

$$0 \leq [\alpha, \epsilon, \nu, \alpha + \epsilon + \nu] \leq \omega - 1. \tag{10}$$

A detailed derivation of (6) and (8) was given by the author in the first article (in Russian with summary in French) of the collection "Ouvrages sur la théory de l'élasticité" fasc. I.

2. **On Three Ways of Solution of Problems of the Theory of Elasticity.** An isotropic solid in equilibrium, as known, must satisfy the four kinds of conditions of: (I) equilibrium, (II) isotropy, (III) loading and (IV) supporting. There are three ways for satisfying these conditions by means of integral algebraical functions: direct, inverse and combined.

The direct way is that when an appropriate value of degree ω is given in the expressions of displacements (2) and equations are formed, according to the conditions I, III and IV, only with the displacement coefficients (3). In this case the conditions of internal equilibrium are expressed by three equations in terms of displacements. These equations may be written, following A. E. H. Love (see A. E. H. Love. A Treatise on the Mathematical Theory of Elasticity, Cambridge, 1927, Chapter V, §91) in the compact form, according to (I)

$$(\lambda + \mu)\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\Delta + \mu\nabla^2[u, v, w] = -q[\mathrm{Cos}\,(q, x), \mathrm{Cos}\,(q, y), \mathrm{Cos}\,(q, z)], \tag{11}$$

where the cubic dilatation is

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{12}$$

and ∇^2 denotes the Laplacian of any function

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{13}$$

These equations were established for an isotropic solid with the two coefficients of elasticity λ and μ by Lamé (see L. Lecornu. Théorie mathématique de l'élasticité, Paris, 1929, p. 3); therefore we call them equations of internal equilibrium of Lamé. Dependences between the displacement coefficients (3) derived from Lamé's equations were given in the above article "Abridged Formulas. . . " for all the coefficients of

second order by the formulas (24) and for those of all orders higher than the second by the formulas (21)–(23).

By determining deformations, rotations and unit stresses in this case, according to fundamental formulas of the theory of elasticity, the conditions of isotropy will be automatically satisfied, which represents a considerable advantage of the direct way.

The inverse way is that when a suitable value of degree $\omega - 1$ is picked out in the expressions of unit stresses (5) and (7), and equations are formed according to the conditions I, II and III only with the stress coefficients

$$H_{\alpha\nu}, K_{\alpha\nu}, L_{\alpha\nu}, P_{\alpha\nu}, Q_{\alpha\nu}, R_{\alpha\nu}. \tag{14}$$

Afterwards, taking into consideration the supporting conditions of the solid under discussion the displacement coefficients (3) are determined in terms of the stress coefficients obtained from the above equations, for which reason this way is called inverse.

This fact may complicate the solution of problems by means of the inverse way. Indeed, some deficiency of indetermined displacement coefficients may take place in this case, and will make it impossible to satisfy all the supporting conditions of the solid. It is evident that in this case the adopted degree $\omega - 1$ of the functions of stresses (5) and (7) is to be increased and all algebraical computations are to be newly performed, if, of course, the method of integral algebraical functions can be applied to the solution of the given problem.

A combined way is that when a suitable value of degree ω is taken in the expressions of displacements (2), and when, according to the above four conditions, we form one portion of equations with displacement coefficients (3) and the other portion with stress coefficients (14). Application of this way enables us to obtain equations with the least number of unknown coefficients.

The number of terms of an integral algebraical function of degree ω with three variables is

$$\tfrac{1}{6}(\omega + 3)(\omega + 2)(\omega + 1). \tag{15}$$

Consequently, the three expressions of displacements (2) give the number of displacement coefficients

$$N_0 = \tfrac{1}{2}(\omega + 3)(\omega + 2)(\omega + 1),$$

and the six formulas (5) and (7) contain

$$N = (\omega + 2)(\omega + 1)\omega$$

of stress coefficients. Hence

$$N - N_0 = \tfrac{1}{2}(\omega + 2)(\omega + 1)(\omega - 3), \tag{16}$$

which gives

$$N > N_0 \text{ when } \omega > 3,$$

$$N = N_0 \text{ when } \omega = 3,$$

$$N < N_0 \text{ when } \omega < 3.$$

Therefore, in most problems when $\omega > 3$ the direct way is preferable as it gives the least number of unknown coefficients.

3. **Expressions of Rotations at the Current Point of a Solid.** As it was expounded in the author's article "Flexure of a Rectangular Covering Slab" . . . (Journal of Mathematics and Physics, vol. XVIII, number 3, 1939), the Theory of Structures treats three degrees of restraint of a plane support section of a member, which are defined by the two analytical conditions: by the displacement of the centre of this section, and by the slope of the tangent to the strained axis at the same centre. The slope of a plane cross section called the deviation of this section and equal to the slope of the strained axis of a member or, in particular, of a beam is a constant for all points of this section, according to the hypothesis of Jacques Bernoulli. In the theory of elasticity this hypothesis of plane sections is justified only in few cases. Consequently, the deviation at points of the same cross section will, in general, be a variable.

In the theory of elasticity it has been proved (see P. Appell. Traité de mécanique rationnelle, tome 3-me, Paris, 1921, Chapitre XXXII, §§679, 685) that the displacement of an elementary vector taken at the current point of a solid is a geometrical sum of its three displacements of: translation, elementary rotation and pure deformation. The angular displacement or elementary rotation of this vector is determined by means of Cauchy's formulas

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{17}$$

which give the elementary rotations at the current point about the axes drawn through it parallel to the coordinate axes X, Y, Z. The subscripts x, y, z of ω indicate the sense of an outward normal to planes on which these rotations are observed.

On the basis of the above abridged formulas of partial derivatives, the expressions of rotations are obtained from (17) in the abridged form

$$\left.\begin{aligned}\omega_x &= \frac{1}{2}\sum_0^{\omega-1} x^\alpha y^\epsilon z^\nu \cdot [(\epsilon+1)C_{\alpha(\epsilon+1)\nu} - (\nu+1)B_{\alpha\epsilon(\nu+1)}],\\ \omega_y &= \frac{1}{2}\sum_0^{\omega-1} x^\alpha y^\epsilon z^\nu \cdot [(\nu+1)A_{\alpha\epsilon(\nu+1)} - (\alpha+1)C_{(\alpha+1)\epsilon\nu}],\\ \omega_z &= \frac{1}{2}\sum_0^{\omega-1} x^\alpha y^\epsilon z^\nu \cdot [(\alpha+1)B_{(\alpha+1)\epsilon\nu} - (\epsilon+1)A_{\alpha(\epsilon+1)\nu}],\end{aligned}\right\} \tag{18}$$

in which the indices α, ϵ, ν get all integral positive values subject to the condition (10).

The formulas (2) and (18) enable us, according to the supporting conditions of the given solid, to form two systems of equations in terms of the displacement coefficients (3). The first system must correspond to the conditions of rectilinear motion and the second to those of rotation at the support points of the solid. These two conditions characterize the degree of restraint of the solid.

Thus, the formulas (2) and (18) are necessary for the complete solution of problems by means of the method of integral algebraical functions in each of the three ways.

4. **On Determining the Positive Direction of Rotation at the Current Point.** In order to state which direction of rotation is to be regarded as positive on each of the three coordinate planes

$$(X, Y), (Y, Z), (Z, X)$$

let us consider the geometrical derivation of the formula III (17) given by professor S. I. Belzecki (Theory of Elasticity, S. Petersburg, 1913, lithographic edition in Russian, p.p. 22–24) supplementing it with explanations from the theory of deformations (see P. Appell. Traité de mécanique rationnelle, tome 3-me, Paris, 1921, Chapitre XXXII, §§663, 669, 674, 680, 681, 683, 685).

In the theory of elasticity it is proved that every element of the infinitesimal volume of a homogeneous elastic solid may be regarded as a homogeneously strained body which is characterized by the fact that its displacements u, v, w are expressed by the linear functions of its initial coordinates. Every cube imaginarily isolated from such a solid is strained into a rhomboidal parallelepiped, i.e., its square faces become rhombs.

For deriving the formula III (17) let us use the principle of superposition of small deformations and consider therefore the deformation of the elementary square $A_0B_0C_0O$ (fig. 1) only on the plane (X, Y).

Every vector drawn in that square would turn through the same angle, if deformation were not accompanied by shear. The deformation of shear, as known, is proportional to the elementary angle by which the right angle is changed.

Let us suppose the diagonal BP of the rhomb is turned to the right through an angle ω_z with respect to the diagonal B_0P of the original square. The diagonals of the square mutually perpendicular become, after deforming, the diagonals of the rhomb in which they are, as

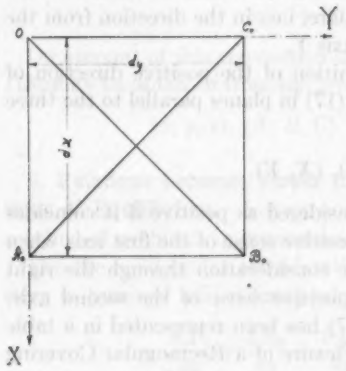

FIG. 1

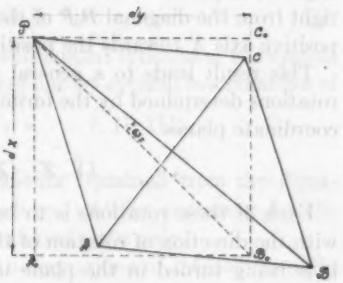

FIG. 2

known, also mutually perpendicular. Consequently, the diagonals of the square at the deformation under consideration are its principal axes, since they do not suffer any deformation of shear. Therefore, the sought expression ω_z is an angular displacement of the diagonal PB_0 or, what is the same thing, a rotation at all points of that elementary square.

Let us denote the displacements of the current point from O (fig. 1) to P (fig. 2) by u, v. Then from the theory of elasticity (see S. Timoschenko. Theory of Elasticity, London, New York, 1934. Chapter V, §45) we have

$$\angle A_0PA = \frac{\partial v}{\partial x}, \qquad \angle C_0PC = \frac{\partial u}{\partial y},$$

and consequently from the fig. 2

$$\angle APC = \frac{\pi}{2} - \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \qquad \angle APB = \angle BPC = \tfrac{1}{2}\angle APC,$$

$$\angle A_0PB_0 = \frac{\pi}{4}$$

and therefore

$$\omega_z = \angle B_0PB = \angle A_0PA + \angle APB - \angle A_0PB_0 = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),$$

which corresponds to the formula III (17). Hence it is seen that the angle ω_z is positive when the diagonal BP of rhomb is turned to the right from the diagonal B_0P of the square, i.e., in the direction from the positive axis X towards the positive axis Y.

This result leads to a general definition of the positive direction of rotations determined by the formulas (17) in planes parallel to the three coordinate planes

$$(Y, Z), (Z, X), (X, Y).$$

Each of these rotations is to be considered as positive if it coincides with the direction of rotation of the positive sense of the first axis when it is being turned in the plane under consideration through the right angle in order to coincide with the positive sense of the second axis. This rule of signs for the rotations (17) has been represented in a table in §2 of the above author's article "Flexure of a Rectangular Covering Slab. . . ."

The formulas II (17) and III (17) may, evidently, be obtained from I (17) by cyclical interchanges of

$$(x, y, z), (u, v, w).$$

5. **On the Cyclical Interchange in the Abridged Formulas.** Constructing the abridged form of an integral algebraical function of the three independent variables x, y, z the following statement is to be observed for the product under the sign of summation: (1) the first variable factor is x, the second y, and the third z, (2) the coefficient subscript contains three letters, the first of them corresponds to the exponent of x, the second to that of y, and the third one of z.

Taking this statement into consideration, it will be seen from the two systems (6) and (8) that a subsequent formula of each system may be

obtained from the preceding formula of the same system by means of cyclical interchanges of

$$(H, K, L), (P, Q, R), (A, B, C), (\alpha, \epsilon, \nu)$$

rearranging the order of the letters in the subscripts of the coefficients A, B, C simultaneously, according to the following symbolic convention

$$(\alpha, \epsilon, \nu, \cdots \text{I, II, III}). \tag{19}$$

Thus, the following coefficient subscripts are substituted

$$(\alpha + 1)\epsilon\nu \text{ by } \alpha(\epsilon + 1)\nu,$$
$$\alpha(\epsilon + 1)\nu \text{ by } \alpha\epsilon(\nu + 1),$$
$$\alpha\epsilon(\nu + 1) \text{ by } (\alpha + 1)\epsilon\nu.$$

On account of this convention each subsequent formula of the system (18) may be obtained from the preceding one by cyclical interchanges of

$$(x, y, z), (A, B, C), (\alpha, \epsilon, \nu, \cdots \text{I, II, III}).$$

6. **Relations between Stress Coefficients Obtained from the Equations of Equilibrium.** Let us now deduce formulas necessary for the solution of problems in inverse and combined ways only.

For the solid under consideration the equations of internal equilibrium in terms of unit stresses take the following form

$$\left.\begin{aligned} \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} &= -q \operatorname{Cos}(q, x), \\ \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} &= -q \operatorname{Cos}(q, y), \\ \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} &= -q \operatorname{Cos}(q, z). \end{aligned}\right\} \tag{20}$$

According to the initial condition, the right members of these equations are constant; therefore the substitution of the corresponding partial derivatives of the unit stresses (5) and (7) into them gives the two following systems: with the stress coefficients of the first order

$$\left.\begin{aligned} H_{100} + R_{010} + Q_{001} &= -q \operatorname{Cos}(q, x), \\ R_{100} + K_{010} + P_{001} &= -q \operatorname{Cos}(q, y), \\ Q_{100} + P_{010} + L_{001} &= -q \operatorname{Cos}(q, z), \end{aligned}\right\} \tag{21}$$

and with the stress coefficients of orders higher than the first

$$\left.\begin{aligned}(\alpha+1)H_{(\alpha+1)\epsilon\nu}+(\epsilon+1)R_{\alpha(\epsilon+1)\nu}+(\nu+1)Q_{\alpha\epsilon(\nu+1)}&=0,\\(\alpha+1)R_{(\alpha+1)\epsilon\nu}+(\epsilon+1)K_{\alpha(\epsilon+1)\nu}+(\nu+1)P_{\alpha\epsilon(\nu+1)}&=0,\\(\alpha+1)Q_{(\alpha+1)\epsilon\nu}+(\epsilon+1)P_{\alpha(\epsilon+1)\nu}+(\nu+1)L_{\alpha\epsilon(\nu+1)}&=0,\end{aligned}\right\}\qquad(22)$$

as the equations obtained must be satisfied by all possible values of the coordinates x, y, z. Evidently the stress coefficients of order zero are not connected by the equations (20).

If only external forces act over the solid, the own weight being disregarded or, which is the same thing, when $q = 0$ the system (21) will, evidently, be identical to the system (22) for the stress coefficients of the first order, as in that case $\alpha = \epsilon = \nu = 0$.

According to the initial condition the partial derivatives of stresses in (20) are integral algebraical functions of degree $\omega - 2$. Therefore the equations (22) are to be applied to all possible combinations of the integral positive indices α, ϵ, ν subject to the conditions

$$0 \leq [\alpha, \epsilon, \nu] \leq \omega - 2, \qquad 1 \leq \alpha + \epsilon + \nu \leq \omega - 2, \qquad (23)$$

since the condition $\alpha + \epsilon + \nu = 0$ in the equations obtained from (20) is expressed by the system (21) as was mentioned above.

As the stress coefficients (14) will also depend on the loading and supporting conditions or, in other words, on boundary conditions of the solid, we conclude from the equations (21) and (22), that some stress coefficients of orders higher than the first may also depend on the magnitude and direction of gravity force.

It will not be difficult to verify according to the convention established in §5 that the formulas II (22) and III (22) may be obtained from I (22) by cyclical interchanges of

$$(H, K, L), (P, Q, R), (\alpha, \epsilon, \nu, \cdots \text{ I, II, III}).$$

7. **Relations between the Stress Coefficients Derived from Beltrami's Equations.** The six functions of stresses (5) and (7), as known from the theory of elasticity, must satisfy, besides the equations of internal equilibrium, also six conditions of isotropy, which are expressed by either the conditions of compatibility of deformations (published by Barré de Saint Venant in 1861), or by those of stresses (given by Beltrami in 1892). We apply Beltrami's equations, as we shall further show that they give simpler dependences between the stress coefficients. In the case under consideration, when according to (I) the volume force is con-

stant, Beltrami's equations take the following form (see S. Timoshenko. Theory of elasticity, London, New York, 1934, Chapter VII, §61, formulas [119]) for normal stresses

$$(1+\sigma)\nabla^2 X_x + \frac{\partial^2 S}{\partial x^2} = 0, \qquad (1+\sigma)\nabla^2 Y_y + \frac{\partial^2 S}{\partial y^2} = 0,$$
$$(1+\sigma)\nabla^2 Z_z + \frac{\partial^2 S}{\partial z^2} = 0, \tag{24}$$

and for tangential stresses

$$(1+\sigma)\nabla^2 X_y + \frac{\partial^2 S}{\partial x \partial y} = 0, \qquad (1+\sigma)\nabla^2 Y_z + \frac{\partial^2 S}{\partial y \partial z} = 0,$$
$$(1+\sigma)\nabla^2 Z_x + \frac{\partial^2 S}{\partial z \partial x} = 0, \tag{25}$$

where the notations:

$$S = X_x + Y_y + Z_z, \tag{26}$$

and (13) are introduced.

We obtain according to (5)

$$S = \sum_{0}^{\alpha-1} (H_{\alpha\nu} + K_{\alpha\nu} + L_{\alpha\nu}) x^\alpha y^\epsilon z^\nu. \tag{27}$$

Substituting the corresponding partial derivatives of the second order from (5) and (27) into the equations (24) and taking into consideration that these equations must be true for all points of the solid, i.e., for all possible combinations of the coordinates x, y, z we obtain the first condition of isotropy from I (24)

$$\begin{aligned}(1+\sigma)[&(\alpha+1)(\alpha+2)H_{(\alpha+2)\epsilon\nu} + (\epsilon+1)(\epsilon+2)H_{\alpha(\epsilon+2)\nu} \\ &+ (\nu+1)(\nu+2)H_{\alpha\epsilon(\nu+2)}] \\ &+ (\alpha+1)(\alpha+2)[H_{(\alpha+2)\epsilon\nu} + K_{(\alpha+2)\epsilon\nu} + L_{(\alpha+2)\epsilon\nu}] = 0,\end{aligned} \tag{28}$$

the second—from II (24)

$$\begin{aligned}(1+\sigma)[&(\alpha+1)(\alpha+2)K_{(\alpha+2)\epsilon\nu} + (\epsilon+1)(\epsilon+2)K_{\alpha(\epsilon+2)\nu} \\ &+ (\nu+1)(\nu+2)K_{\alpha\epsilon(\nu+2)}] \\ &+ (\epsilon+1)(\epsilon+2)[H_{\alpha(\epsilon+2)\nu} + K_{\alpha(\epsilon+2)\nu} + L_{\alpha(\epsilon+2)\nu}] = 0,\end{aligned} \tag{29}$$

and the third—from III (24)

$$(1+\sigma)[(\alpha+1)(\alpha+2)L_{(\alpha+2)\omega}+(\epsilon+1)(\epsilon+2)L_{\alpha(\epsilon+2)\nu}+(\nu+1)(\nu+2)L_{\alpha\epsilon(\nu+2)}]+(\nu+1)(\nu+2)[H_{\alpha\epsilon(\nu+2)}+K_{\alpha\epsilon(\nu+2)}+L_{\alpha\epsilon(\nu+2)}]=0. \quad (30)$$

Substituting, similarly, the corresponding derivatives from (7) and (27) into the system (25), we find the fourth condition of isotropy from I (25)

$$(1+\sigma)[(\alpha+1)(\alpha+2)R_{(\alpha+2)\epsilon\nu}+(\epsilon+1)(\epsilon+2)R_{\alpha(\epsilon+2)\nu}+(\nu+1)(\nu+2)R_{\alpha\epsilon(\nu+2)}]+(\alpha+1)(\epsilon+1)[H_{(\alpha+1)(\epsilon+1)\nu}+K_{(\alpha+1)(\epsilon+1)\nu}+L_{(\alpha+1)(\epsilon+1)\nu}]=0, \quad (31)$$

the fifth—from II (25)

$$(1+\sigma)[(\alpha+1)(\alpha+2)P_{(\alpha+2)\epsilon\nu}+(\epsilon+1)(\epsilon+2)P_{\alpha(\epsilon+2)\nu}+(\nu+1)(\nu+2)P_{\alpha\epsilon(\nu+2)}]+(\epsilon+1)(\nu+1)[H_{\alpha(\epsilon+1)(\nu+1)}+K_{\alpha(\epsilon+1)(\nu+1)}+L_{\alpha(\epsilon+1)(\nu+1)}]=0, \quad (32)$$

and the sixth—from III (25)

$$(1+\sigma)[(\alpha+1)(\alpha+2)Q_{(\alpha+2)\epsilon\nu}+(\epsilon+1)(\epsilon+2)Q_{\alpha(\epsilon+2)\nu}+(\nu+1)(\nu+2)Q_{\alpha\epsilon(\nu+2)}]+(\nu+1)(\alpha+1)[H_{(\alpha+1)\epsilon(\nu+1)}+K_{(\alpha+1)\epsilon(\nu+1)}+L_{(\alpha+1)\epsilon(\nu+1)}]=0. \quad (33)$$

Having in view the symbolic convention of §5 we may convince ourselves that every subsequent formula of the system (28)–(30) is obtained by cyclical interchanges of

$$(H, K, L), \qquad (\alpha, \epsilon, \nu, \cdots I, II, III)$$

and that of the system (31)–(33) by cyclical interchanges of

$$(P, Q, R), \qquad (\alpha, \epsilon, \nu, \cdots I, II, III)$$

from the preceding formula of the corresponding system.

The six conditions of isotropy (28)–(33) have been obtained as a result of twofold differentiation of the functions (5), (7) and (27) which are integral algebraical functions of degree $\omega - 1$. Consequently the six conditions of isotropy are to be applied to all possible combinations of the integral positive indices α, ϵ, ν according to the condition

$$0 \leq [\alpha, \epsilon, \nu, \alpha + \epsilon + \nu] \leq \omega - 3. \tag{34}$$

Each of the formulas (28)–(33) contains six stress coefficients. If the conditions of isotropy are obtained from Saint Venant's equations of compatibility of deformations (see S. Timoshenko, id. formulas [118]) then the first three conditions will contain seven stress coefficients each and the second three conditions—six stress coefficients each. Thus Beltrami's equations represent a certain simplification in deriving the conditions of isotropy.

8. **Determination of Displacement Coefficients on the Basis of Stress Coefficients.** Having obtained all the stress coefficients (14) according to: (1) the equations (21) and (22), (2) the conditions of isotropy (28)–(33) and (3) given boundary conditions, the solution of a given problem of the theory of elasticity cannot be considered as complete. Yet it is necessary to verify whether the obtained functions of stresses (5) and (7) correspond to the supporting conditions of the solid. For that purpose we must have the expressions both of the displacements (2) and rotations (18) or in other words find the displacement coefficients (3).

Let us solve the system (6) for the displacement coefficients

$$A_{(\alpha+1)\sigma}, \qquad B_{\alpha(\epsilon+1)\sigma}, \qquad C_{\alpha\epsilon(\nu+1)}.$$

The fundamental determinant of that system

$$4\mu^2(2\mu + 3\lambda)(\alpha + 1)(\epsilon + 1)(\nu + 1)$$

cannot be equal to zero and consequently the system (6) always gives finite values of the sought coefficients. Passing, according to (9) from the coefficients λ, μ to E, σ the result of the solution of three equations (6) may be expressed by the system:

$$\frac{(\alpha + 1)A_{(\alpha+1)\sigma}}{\sigma L_{\alpha\sigma} - H_{\alpha\sigma} + \sigma K_{\alpha\sigma}} = \frac{(\epsilon + 1)B_{\alpha(\epsilon+1)\sigma}}{\sigma H_{\alpha\sigma} - K_{\alpha\sigma} + \sigma L_{\alpha\sigma}}$$

$$= \frac{(\nu + 1)C_{\alpha\epsilon(\nu+1)}}{\sigma K_{\alpha\sigma} - L_{\alpha\sigma} + \sigma H_{\alpha\sigma}} = -\frac{1}{E}. \tag{35}$$

In the formulas (6) the indices α, ϵ, ν take all positive values according to the condition (10).

Therefore the system (35) must be applied to all possible combinations according to the same condition (10).

Let us now transform the system (8): the index ν in the first equation

is substituted by $\nu + 1$, the index α in the second—by $\alpha + 1$, the index ϵ in the third—by $\epsilon + 1$, the value of the coefficient of elasticity μ being substituted in the obtained equations by II (9). Then we get the system

$$(\epsilon + 1)A_{\alpha(\epsilon+1)(\nu+1)} + (\alpha + 1)B_{(\alpha+1)\epsilon(\nu+1)} = 2\frac{1+\sigma}{E}R_{\alpha\epsilon(\nu+1)},$$

$$(\nu + 1)B_{(\alpha+1)\epsilon(\nu+1)} + (\epsilon + 1)C_{(\alpha+1)(\epsilon+1)\nu} = 2\frac{1+\sigma}{E}P_{(\alpha+1)\epsilon\nu},$$

$$(\alpha + 1)C_{(\alpha+1)(\epsilon+1)\nu} + (\nu + 1)A_{\alpha(\epsilon+1)(\nu+1)} = 2\frac{1+\sigma}{E}Q_{\alpha(\epsilon+1)\nu},$$

whose fundamental determinant

$$2(\alpha + 1)(\epsilon + 1)(\nu + 1)$$

is also not equal to zero. For the sake of simplicity, we also represent the results of solving these equations by the system:

$$\frac{(\epsilon + 1)(\nu + 1)A_{\alpha(\epsilon+1)(\nu+1)}}{(\nu + 1)R_{\alpha\epsilon(\nu+1)} - (\alpha + 1)P_{(\alpha+1)\epsilon\nu} + (\epsilon + 1)Q_{\alpha(\epsilon+1)\nu}}$$

$$= \frac{(\nu + 1)(\alpha + 1)B_{(\alpha+1)\epsilon(\nu+1)}}{(\alpha + 1)P_{(\alpha+1)\epsilon\nu} - (\epsilon + 1)Q_{\alpha(\epsilon+1)\nu} + (\nu + 1)R_{\alpha\epsilon(\nu+1)}}$$

$$= \frac{(\alpha + 1)(\epsilon + 1)C_{(\alpha+1)(\epsilon+1)\nu}}{(\epsilon + 1)Q_{\alpha(\epsilon+1)\nu} - (\nu + 1)R_{\alpha\epsilon(\nu+1)} + (\alpha + 1)P_{(\alpha+1)\epsilon\nu}} = \frac{1+\sigma}{E}. \quad (36)$$

Applying (35) and (36) in due limits for all possible combinations of the integral positive indices α, ϵ, ν without exception, we shall evidently evaluate some of the displacement coefficients (3) twice, but we shall still not get all displacement coefficients which might be determined by the sets (6) and (8). Wanting displacement coefficients may be obtained by solving the system (8) together with the found formulas (35) having in view II (9). Substituting the index ϵ in I (8) by $\epsilon + 1$, and the index α in II (35) by $\alpha + 1$, and then eliminating the coefficient $B_{(\alpha+1)(\epsilon+1)\nu}$ between the two equations thus obtained—we find

$$E\cdot(\epsilon + 1)(\epsilon + 2)A_{\alpha(\epsilon+2)\nu} = 2(1 + \sigma)(\epsilon + 1)R_{\alpha(\epsilon+1)\nu}$$
$$+ (\alpha + 1)[\sigma H_{(\alpha+1)\epsilon\nu} - K_{(\alpha+1)\epsilon\nu} + \sigma L_{(\alpha+1)\epsilon\nu}]. \quad (37)$$

Similarly we obtain by eliminating the following coefficients: $C_{(\alpha+1)\epsilon(\nu+1)}$ between III (8) and III (35)

$$E\cdot(\nu+1)(\nu+2)A_{\alpha\epsilon(\nu+2)} = 2(1+\sigma)(\nu+1)Q_{\alpha\epsilon(\nu+1)} + (\alpha+1)[\sigma K_{(\alpha+1)\epsilon\nu} - L_{(\alpha+1)\epsilon\nu} + \sigma H_{(\alpha+1)\epsilon\nu}], \quad (38)$$

$C_{\alpha(\epsilon+1)(\nu+1)}$ between II (8) and III (35)

$$E\cdot(\nu+1)(\nu+2)B_{\alpha\epsilon(\nu+2)} = 2(1+\sigma)(\nu+1)P_{\alpha\epsilon(\nu+1)} + (\epsilon+1)[\sigma K_{\alpha(\epsilon+1)\nu} - L_{\alpha(\epsilon+1)\nu} + \sigma K_{\alpha(\epsilon+1)\nu}], \quad (39)$$

$A_{(\alpha+1)(\epsilon+1)\nu}$ between I (8) and I (35)

$$E\cdot(\alpha+1)(\alpha+2)B_{(\alpha+2)\epsilon\nu} = 2(1+\sigma)(\alpha+1)R_{(\alpha+1)\epsilon\nu} + (\epsilon+1)[\sigma L_{\alpha(\epsilon+1)\nu} - H_{\alpha(\epsilon+1)\nu} + \sigma K_{\alpha(\epsilon+1)\nu}], \quad (40)$$

$A_{(\alpha+1)\epsilon(\nu+1)}$ between III (8) and I (35)

$$E\cdot(\alpha+1)(\alpha+2)C_{(\alpha+2)\epsilon\nu} = 2(1+\sigma)(\alpha+1)Q_{(\alpha+1)\epsilon\nu} + (\nu+1)[\sigma L_{\alpha\epsilon(\nu+1)} - H_{\alpha\epsilon(\nu+1)} + \sigma K_{\alpha\epsilon(\nu+1)}], \quad (41)$$

and finally $B_{\alpha(\epsilon+1)(\nu+1)}$ between II (8) and II (35)

$$E\cdot(\epsilon+1)(\epsilon+2)C_{\alpha(\epsilon+2)\nu} = 2(1+\sigma)(\epsilon+1)P_{\alpha(\epsilon+1)\nu} + (\nu+1)[\sigma H_{\alpha\epsilon(\nu+1)} - K_{\alpha\epsilon(\nu+1)} + \sigma L_{\alpha\epsilon(\nu+1)}]. \quad (42)$$

The expressions (36)–(42) for the last displacement coefficients were obtained by means of the formulas (8) and (35), in which the order of all coefficients had been raised by one unit, which corresponds to a onefold differentiation of each of these formulas. Consequently, the integral positive indices α, ϵ, ν in the formulas (36)–(42) are subject to the condition,

$$0 \leq [\alpha, \epsilon, \nu, \alpha+\epsilon+\nu] \leq \omega - 2, \quad (43)$$

since they were connected in the original system (8) by the condition (10).

Thus the two systems (35) and (36) consisting of three formulas each have been obtained for some displacement coefficients and the six formulas (37)–(42) for other displacement coefficients may be represented also in the form of the two following systems

(37), (39), (41)

and (38), (40), (42).

Then, as follows according to the convention (19) every subsequent formula of these four systems may be obtained from the preceding formula of the same system by cyclical interchanges of

$$(A, B, C), \quad (H, K, L), \quad (P, Q, R), \quad (\alpha, \epsilon, \nu, \cdots I, II, III).$$

Comparing the systems (6) and (8) we may see that the subscripts of corresponding displacement coefficients do not coincide although their order is one and the same. Hence we conclude that three displacement coefficients of the first order remain indefinite however the equations of the two systems be combined. Therefore putting $\alpha = \epsilon = \nu = 0$ in the system (8) we may obtain a set of formulas for the six displacement coefficients of the first order

$$\frac{A_{010}}{R_{000} + C} = \frac{A_{001}}{Q_{000} - B} = \frac{B_{001}}{P_{000} + A} = \frac{B_{100}}{R_{000} - C}$$

$$= \frac{C_{100}}{Q_{000} + B} = \frac{C_{010}}{P_{000} - A} = \frac{1 + \sigma}{E}, \quad (44)$$

A, B, C being indefinite constants corresponding to the constants e, c, b of professor S. P. Timoshenko (id, Chapter VII, §62, formulas [b]).

These constants together with the displacement coefficients of order zero

$$A_{000}, \quad B_{000}, \quad C_{000}$$

are to be determined according to the supporting conditions of the body.

If solving a problem in the inverse way we determine all the stress coefficients entirely, these six indefinite constants may prove insufficient to satisfy all supporting conditions of the body. This case has been considered in §2.

The three displacement coefficients of the first order

$$A_{100}, \quad B_{010}, \quad C_{001}$$

may be perfectly determined by (35), if the three coefficients of normal stresses of order zero are known.

9. **On Some Other Relations between Stress Coefficients.** To verify the found values of stress coefficients it is necessary to have still other relations between stress coefficients, taking into consideration the initial condition (1).

From the theory of elasticity it is known that the function S determined by (27) is harmonic in this case (see A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, Cambridge, 1927, §92), that

may also be easily verified by summing up the three Beltrami's equations (24). Using for the harmonic function S the formula (12) given in the author's article "Abridged Formulas . . . ," it will be easy to establish a relation between the coefficients of normal stresses.

As shown by Lamé (Lamé. Leçons sur la théorie de l'élasticité des corps solides, Paris, 1866, §27) each of the stresses (5) and (7) must satisfy a biharmonic equation, that gives six equations. On the basis of these equations called therefore Lamé's conditions (Pigeaud, G. Résistance des matériaux et élasticité, Paris, 1928, pp. 512–513), relations for the coefficients of each of six stresses may be obtained according to the formula (15) of the same article "Abridged Formulas. . . ."

On the basis of the equations of equilibrium (20) a new relation between coefficients of six different stresses can be easily obtained. For that purpose let us take the partial derivatives: of I (20) as to x, of II (20) as to y, of III (20) as to z and sum up the results. Then, according to (5) and (7), using abridged formulas of differentiation of integral algebraical functions, we obtain

$$2(\alpha+1)(\epsilon+1)R_{(\alpha+1)(\epsilon+1)\nu} + 2(\epsilon+1)(\nu+1)P_{\alpha(\epsilon+1)(\nu+1)} + 2(\nu+1)(\alpha+1)Q_{(\alpha+1)\epsilon(\nu+1)} + (\alpha+1)(\alpha+2)H_{(\alpha+2)\epsilon\nu} + (\epsilon+1)(\epsilon+2)K_{\alpha(\epsilon+2)\nu} + (\nu+1)(\nu+2)L_{\alpha\epsilon(\nu+2)} = 0. \quad (45)$$

10. **Some Relations between Displacement Coefficients.** All below mentioned algebraical relations between displacement coefficients (3) derived under the condition (1) can serve for verifying the obtained values of these coefficients.

As shown by Lamé (Lamé, id. §27), the cubic dilatation Δ at the current point of a body defined by (12) represents a harmonic function. The relations (17) are also harmonic functions (see W. J. Ibbetson. An Elementary Treatise on the Mathematical Theory of Perfectly Elastic Solids, London, 1887, §301). Consequently, using the formula (12) of the article "Abridged Formulas . . . " and having the expressions for the displacements (2) and rotations (18), four algebraic relations between displacement coefficients may be formed on account of the conditions above.

As shown by Lamé (Lamé, id. §27), each of the displacements (2) is a biharmonic function. Therefore, three algebraical relations with displacement coefficients A, B, C respectively may be formed by means of the formula (15) of the aricle "Abridged Formulas. . . ."

HARBIN.

SAINT VENANT'S CONDITIONS OF COMPATIBILITY IN THE METHOD OF INTEGRAL ALGEBRAICAL FUNCTIONS

By S. A. Savin

1. **Transformation of Saint Venant's conditions of compatibility.** Introducing the following notations:

λ_x, λ_y, λ_z—linear unit deformations, that is unit elongations or unit contractions in the direction of the axes X, Y, Z respectively,

γ_x, γ_y, γ_z—shearing deformations or shear strains in the planes perpendicular to the axes X, Y, Z respectively, Saint Venant's conditions of compatibility of deformations are expressed by the two symmetrical systems (see S. Timoshenko. Theory of Elasticity, London, New York 1934, Chapter VII, Paragraph 61, formulas [118]):

$$\left.\begin{aligned} \frac{\partial^2 \gamma_x}{\partial x \partial y} &= \frac{\partial^2 \lambda_y}{\partial x^2} + \frac{\partial^2 \lambda_x}{\partial y^2}, \\ \frac{\partial^2 \gamma_x}{\partial y \partial z} &= \frac{\partial^2 \lambda_x}{\partial y^2} + \frac{\partial^2 \lambda_y}{\partial z^2}, \\ \frac{\partial^2 \gamma_y}{\partial z \partial x} &= \frac{\partial^2 \lambda_x}{\partial z^2} + \frac{\partial^2 \lambda_z}{\partial x^2}, \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} 2\frac{\partial^2 \lambda_x}{\partial y \partial z} &= \frac{\partial}{\partial x}\left[\frac{\partial \gamma_x}{\partial z} - \frac{\partial \gamma_z}{\partial x} + \frac{\partial \gamma_y}{\partial y}\right], \\ 2\frac{\partial^2 \lambda_y}{\partial z \partial x} &= \frac{\partial}{\partial y}\left[\frac{\partial \gamma_z}{\partial x} - \frac{\partial \gamma_y}{\partial y} + \frac{\partial \gamma_x}{\partial z}\right], \\ 2\frac{\partial^2 \lambda_z}{\partial x \partial y} &= \frac{\partial}{\partial z}\left[\frac{\partial \gamma_y}{\partial y} - \frac{\partial \gamma_x}{\partial z} + \frac{\partial \gamma_z}{\partial x}\right]. \end{aligned}\right\} \tag{2}$$

Hence it may be seen that a subsequent formula of each system can be obtained from the preceding one of the same system by means of cyclical interchanges of x, y, z.

The theory of elasticity gives (see S. Timoshenko, id., Paragraph 6) the expressions for linear unit deformations

$$\left.\begin{aligned} E\cdot\lambda_x &= -(\sigma Z_z - X_x + \sigma Y_y), \\ E\cdot\lambda_y &= -(\sigma X_x - Y_y + \sigma Z_z), \\ E\cdot\lambda_z &= -(\sigma Y_y - Z_z + \sigma X_x), \end{aligned}\right\} \tag{3}$$

and for shearing deformations

$$\frac{\gamma_x}{Y_x} = \frac{\gamma_y}{Z_y} = \frac{\gamma_z}{X_z} = \frac{2(1+\sigma)}{E} \tag{4}$$

in terms of the normal unit stresses

$$X_x, Y_y, Z_z, \tag{5}$$

and of the tangential or shearing unit stresses

$$X_y = Y_x, Y_z = Z_y, Z_x = X_z, \tag{6}$$

where E denotes Young's modulus and σ Poisson's ratio of transverse deformation.

According to (3) and (4), the system (1) is transformed as follows

$$\left.\begin{aligned} &2(1+\sigma)\frac{\partial^2 X_y}{\partial x \partial y} + \frac{\partial^2}{\partial x^2}[\sigma X_x - Y_y + \sigma Z_z] + \frac{\partial^2}{\partial y^2}[\sigma Z_z - X_x + \sigma Y_y] = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \end{aligned}\right\} \tag{7}$$

and the system (2) is obtained in the simpler form

$$\left.\begin{aligned} &\frac{\partial^2}{\partial y \partial z}[\sigma Z_z - X_x + \sigma Y_y] + (1+\sigma)\frac{\partial}{\partial x}\left[\frac{\partial X_y}{\partial z} - \frac{\partial Y_z}{\partial x} + \frac{\partial Z_x}{\partial y}\right] = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \end{aligned}\right\} \tag{8}$$

The two formulas not written in each of these systems may be obtained by the cyclical interchanges of (x, y, z) and (X, Y, Z).

2. Saint Venant's conditions in terms of stress coefficients. The expressions of displacements and stresses by means of integral algebraical functions have been given in the author's article "Method of Integral Algebraical Functions in the Theory of Elasticity of Three Dimensions" (Journal of Mathematics and Physics, vol. XX, number 1, pp. 1–17) by the formulas (2), (5) and (7).

In this case the displacements u, v, w along the coordinate axes X, Y, Z are expressed in the symbolic form

$$[u; v; w] = \sum_0^\infty [A_{\alpha\alpha'}; B_{\alpha\alpha'}; C_{\alpha\alpha'}]\, x^\alpha y^{\alpha'} z^{\alpha'}. \tag{9}$$

Then the expressions of stresses can also be represented in the compact forms: of the normal unit stresses

$$[X_x;\ Y_y;\ Z_z] = \sum_{0}^{\omega-1} [H_{\alpha\nu};\ K_{\alpha\nu};\ L_{\alpha\nu}]\, x^{\alpha} y^{\nu} z^{\nu}, \tag{10}$$

and of the tangential unit stresses

$$[X_y;\ Y_z;\ Z_x] = \sum_{0}^{\omega-1} [R_{\alpha\nu};\ P_{\alpha\nu};\ Q_{\alpha\nu}]\, x^{\alpha} y^{\nu} z^{\nu}. \tag{11}$$

In the formulas (9)–(11)

$$A,\ B,\ C, \tag{12}$$

are displacement coefficients, and

$$H,\ K,\ L,\ P,\ Q,\ R \tag{13}$$

—stress coefficients.

Using the abridged formulas of differentiation of integral algebraical functions given in the author's article "Abridged Formulas for the Solution of Differential Equations of the Theory of Elasticity by means of Integral Polynomials" (Journal of Mathematics and Physics, vol. XVII, n. 4, January 1939), and knowing that the equations (7) and (8) must be satisfied by all possible values of the coordinates x, y, z of any isotropic body under consideration, we obtain according to (10) and (11) the conditions: from the system (7)

$$\left.\begin{aligned} &2(1+\sigma)(\alpha+1)(\varepsilon+1)R_{(\alpha+1)(\varepsilon+1)\nu} \\ &\quad + (\alpha+1)(\alpha+2)[\sigma H_{(\alpha+2)\varepsilon\nu} - K_{(\alpha+2)\varepsilon\nu} + \sigma L_{(\alpha+2)\varepsilon\nu}] \\ &\quad + (\varepsilon+1)(\varepsilon+2)[\sigma L_{\alpha(\varepsilon+2)\nu} - H_{\alpha(\varepsilon+2)\nu} + \sigma K_{\alpha(\varepsilon+2)\nu}] = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \end{aligned}\right\} \tag{14}$$

and from the system (8)

$$\left.\begin{aligned} &(\varepsilon+1)(\nu+1)[\sigma L_{\alpha(\varepsilon+1)(\nu+1)} - H_{\alpha(\varepsilon+1)(\nu+1)} + \sigma K_{\alpha(\varepsilon+1)(\nu+1)}] \\ &\quad + (1+\sigma)(\alpha+1)[(\nu+1)R_{(\alpha+1)\varepsilon(\nu+1)} - (\alpha+2)P_{(\alpha+2)\varepsilon\nu} \\ &\quad + (\varepsilon+1)Q_{(\alpha+1)(\varepsilon+1)\nu}] = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{aligned}\right\} \tag{15}$$

The second and third formulas not written in the systems (14) and (15) may be obtained by means of the cyclical interchanges of

$$(H, K, L), (P, Q, R), (\alpha, \epsilon, \nu, \cdots \text{I, II, III}),$$

bearing in mind the symbolical convention for the circular permutation of the subscripts α, ϵ, ν given in the paragraph 5 of the article "Method of Integral Algebraical Functions ··· ", which we shall further call simply "the preceding article".

3. **Another possible way for obtaining the systems** (14) **and** (15). The formulas (14) and (15) represent the six conditions of isotropy derived from Saint Venant's conditions of compatibility of deformations. These conditions of isotropy may also be directly obtained by means of the formulas given in the paragraph 8 of "the preceding article."

For that purpose it is necessary to determine the expressions of the same displacement coefficient of the order $\alpha + \epsilon + \nu + 3$ by two different formulas of "the preceding article" and then equate them as it is shown in the table below.

Displacement coefficients	Determined by the formulas	Obtained isotropy conditions
$A_{(\alpha+1)(\epsilon+2)\nu}$	(35), (37)	I (14)
$B_{\alpha(\epsilon+1)(\nu+2)}$	(35), (39)	II (14)
$C_{(\alpha+2)\epsilon(\nu+1)}$	(35), (41)	III (14)
$A_{(\alpha+1)(\epsilon+1)(\nu+1)}$	(35), (36)	I (15)
$B_{(\alpha+1)(\epsilon+1)(\nu+1)}$	(35), (36)	II (15)
$C_{(\alpha+1)(\epsilon+1)(\nu+1)}$	(35), (36)	III (15)

It is easy to verify that no new relations between stress coefficients different from the relations obtained according to (14) and (15) of the present article can be got by means of any other combinations of the formulas (35)–(42) of "the preceding article."

4. **On the two forms of the conditions of isotropy.** The two systems (14) and (15) represent the second form of the conditions of isotropy. The first form obtained from Beltrami's equations has been given by the formulas (28)–(33) of "the preceding article."

From deriving Beltrami's equations (see S. Timoshenko, id., Chapter VII, Par. 61) it is seen that some stress coefficients in the first form of the conditions of isotropy are subject to the equations of internal

equilibrium of the body, which is absent in the second form of those conditions.

Since both the first and second forms of the conditions of isotropy are obtained by a twofold differentiation of the integral algebraical functions (10) and (11) of degree $\omega - 1$, consequently the six conditions of isotropy taken either in the first or second forms must be applied to all possible combinations of the integral positive indices α, ϵ, ν according to the condition

$$0 \leq [\alpha, \epsilon, \nu, \alpha + \epsilon + \nu] \leq \omega - 3. \tag{16}$$

The conditions of isotropy are applied to the solution of a problem of the theory of elasticity by means of inverse or combined way only. Of the two above forms of the conditions of isotropy it is necessary to use that which furnishes the problem under consideration with equations of the least number of unknown stress coefficients.

5. **On the absence of the conditions of isotropy in the direct way of the solution of problems of the theory of elasticity.** In the paragraph 2 of "the preceding article" there is mentioned that the six conditions of isotropy are unnecessary in the direct way of the solution of a problem.

In fact, if the conditions of isotropy given either by the two systems (14) and (15) of the present article or by the formulas (28)–(33) of "the preceding article" are expressed in terms of the displacement coefficients A, B, C, these six conditions become identical. For that purpose the formulas (6) and (8) of "the preceding article" giving the stress coefficients in terms of displacement coefficients are to be used.

HARBIN.

ON SOME SOLUTIONS OF THE EQUATIONS OF INTERNAL EQUILIBRIUM OF THE THEORY OF ELASTICITY

By S. A. Savin

1. **The two ways of the solution of Cauchy and Lamé's equations.** The object of the present article is a consideration of such solutions of the three equations of internal equilibrium which can be applied in technical problems of the theory of elasticity only when the solid under consideration is free from the action of volume forces.

The equations of internal equilibrium taken according to Lamé in terms of the rotations ω_x, ω_y, ω_z (see Lamé. Leçons sur la théorie de l'élasticité des corps solides, Paris, 1866, paragraph 26, p. 67) are not employed in the applied mechanics; therefor we will further consider only two other forms of these equations which have been given in the author's article "Method of Integral Algebraical Functions in the Theory of Elasticity of Three Dimensions" (Journal of Mathematics and Physics, vol. XX, number 1, pp. 1–17). The first form necessary for the solution of problems of the theory of elasticity in inverse and combined ways represents a system of equations in terms of the unit stresses

$$X_x, Y_y, Z_z; X_y = Y_x, Y_z = Z_y, Z_x = X_z. \tag{1}$$

The second form is required for the solution of those problems in direct way and gives a system of equations in terms of the displacements

$$u, v, w \tag{2}$$

along the coordinate axes X, Y, Z respectively.

The first form has been given by the system (20), the second—in a compact form—by the formula (11) of the above article for the case when a homogeneous isotropic solid is in equilibrium under the influence of external forces on its surface and own weight. In those formulas q is the own weight per unit volume; since there are no volume forces in this case, consequently it is necessary to put $q = 0$.

We shall call the first form of the equations of equilibrium as "Cauchy's equations" (see Résumé des leçons données a l'École des Ponts et Chaussées sur l'application de la mécanique, par Navier avec des notes et des appendices par Barré de Saint-Venant, Paris, 1864, appendice 111-me, p. 585) and the second as "Lamé's equations" (see

L. Lecornu. Théorie mathématique de l'élasticité, Paris, 1929, p. 3), i.e. after the above French academicians' names who first published them.

These equations may be solved either by means of direct way, when the sought for values (1) or (2) are expressed by arbitrary functions of the coordinates x, y, z, or indirect way, when the same sought for values either depend only on derivatives of some functions or are given by formulas with arbitrary functions and their derivatives.

It is to be noticed that the solution of this problem in indirect way can give several general solutions as it will further be observed.

The influence of the own weight can be determined separately on the basis of the prinicple of superposition of actions. The displacements u, v, w in this case are expressed by integral algebraical functions. In accordance with Lamé's equations of equilibrium the displacement coefficients must be connected by relations, which are given in the author's article "Abridged Formulas for the Solution of Differential Equations of the Theory of Elasticity by means of Integral Polynomials" (Journal of Mathematics and Physics, vol. XVII, number 4, January 1939) by the formulas (24) for the displacement coefficients of the second order and by the formulas (21)–(23) for the displacement coefficients of all orders higher than the second. The displacement coefficients of the orders zero and first remain perfectly arbitrary.

2. **The simpler solutions of Cauchy's equations of equilibrium.** The first solution of Cauchy's equations was published, as known, by English astronomer G. B. Airy in 1862 (see A. E. H. Love. A Treatise of the Mathematical Theory of Elasticity, Cambridge, 1934, paragraph 56, p. 88 and paragraph 144, p. 204) for a plane problem. This solution is obtained in indirect way: the stresses are determined in terms of the partial derivatives of the second order of an arbitrary biharmonic function of two dimensions which is called "Airy's function."

For the solution of three-dimensional problems in some particular cases, when the influence of the own weight is not taken into account, the potential function of displacements may be applied (see A. E. H. Love, id. paragraph 8, p. 38). Then, denoting this function by ϑ, the following formulas are to be used: for normal unit stresses

$$X_x = \frac{\partial^2 \vartheta}{\partial x^2} + C, \qquad Y_y = \frac{\partial^2 \vartheta}{\partial y^2} + C, \qquad Z_z = \frac{\partial^2 \vartheta}{\partial z^2} + C, \tag{3}$$

and for tangential unit stresses

$$X_y = Y_x = \frac{\partial^2 \vartheta}{\partial x \partial y}, \qquad Y_z = Z_y = \frac{\partial^2 \vartheta}{\partial y \partial z}, \qquad Z_x = X_z = \frac{\partial^2 \vartheta}{\partial z \partial x}, \tag{4}$$

where C is one and the same arbitrary constant and the function $\vartheta(x, y, z)$ must satisfy Poisson's equation

$$\nabla^2 \vartheta(x, y, z) = \text{Const.} \tag{5}$$

It is easy to verify that the substitution of the expressions (3) and (4) into Cauchy's equations of equilibrium makes them identical at the conditions: (5) and $q = 0$.

Having according to (3) and (5) the expression

$$S = X_x + Y_y + Z_z = \nabla^2 \vartheta + 3C = \text{Const.},$$

and knowing that

$$\nabla^2 \frac{\partial^2 \vartheta}{\partial x^2} = \frac{\partial^2}{\partial x^2} \nabla^2 \vartheta, \ \ldots$$

$$\nabla^2 \frac{\partial^2 \vartheta}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} \nabla^2 \vartheta, \ \ldots$$

it will not be difficult to verify that the stresses (3) and (4) also satisfy Beltrami's equations, which have been given by the formulas (24) and (25) of the article "Method of Integral Algebraical Functions. . . ."

The more general solution of Cauchy's equations of equilibrium was given by B. G. Galerkin (see Comptes rendus de l'Académie des sciences de l'URSS, 1931, No. 10 p. 281–286) by means of three arbitrary biharmonic functions.

Introducing the notation

$$\theta = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z} \tag{6}$$

B. G. Galerkin's formulas may be written in the following symmetric form

$$\left.\begin{aligned} X_x &= \left(\sigma \cdot \nabla^2 - \frac{\partial^2}{\partial x^2}\right)\theta + 2(1-\sigma)\frac{\partial}{\partial x}\nabla^2 \psi_1, \\ Y_y &= \left(\sigma \cdot \nabla^2 - \frac{\partial^2}{\partial y^2}\right)\theta + 2(1-\sigma)\frac{\partial}{\partial y}\nabla^2 \psi_2, \\ Z_z &= \left(\sigma \cdot \nabla^2 - \frac{\partial^2}{\partial z^2}\right)\theta + 2(1-\sigma)\frac{\partial}{\partial z}\nabla^2 \psi_3, \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} X_y = Y_x &= (1-\sigma)\cdot\nabla^2\left(\frac{\partial\psi_1}{\partial y}+\frac{\partial\psi_2}{\partial x}\right)-\frac{\partial^2\theta}{\partial x\,\partial y},\\ Y_z = Z_y &= (1-\sigma)\cdot\nabla^2\left(\frac{\partial\psi_2}{\partial z}+\frac{\partial\psi_3}{\partial y}\right)-\frac{\partial^2\theta}{\partial y\,\partial z},\\ Z_x = X_z &= (1-\sigma)\cdot\nabla^2\left(\frac{\partial\psi_3}{\partial x}+\frac{\partial\psi_1}{\partial z}\right)-\frac{\partial^2\theta}{\partial z\,\partial x}, \end{aligned}\right\} \tag{8}$$

where biharmonic functions are denoted by ψ, which gives the condition

$$\nabla^2\nabla^2\psi(x, y, z) = 0. \tag{9}$$

When the own weight of an isotropic body is not taken into consideration the stresses determined by these formulas satisfy Cauchy and Beltrami's equations.

It is to be noted that B. G. Galerkin first published his solution in more complicated form in 1930 (see Comptes rendus des séances de l'Académie de sciences, tome 190, No. 18, Paris, 1930).

Thus B. G. Galerkin's solution has also been obtained by means of indirect way. The solution of Cauchy's equations of equilibrium in direct way has not been found.

3. **Some solutions of Lamé's equations of equilibrium when volume forces are absent.** An essay to solve Lamé's equations of equilibrium in direct way was made by the author in 1926 (see Ouvrages sur la théorie de l'élasticité, fasc. 1, 1934, the fifth article in Russian with an English summary) by means of six arbitrary functions of complex arguments.

Since the system of Lamé's equations of equilibrium consists of the three partial differential equations of the second order with the three unknown functions u, v, w, each displacement is expressed by means of two arbitrary functions, one of which is of the argument $m + jn$ and the other—of the argument $m - jn$ conjugated with the first one; in these arguments

$$m = a_1x + b_1y + c_1z,\ n = a_2x + b_2y + c_2z,\ j = \sqrt{-1}$$

the indeterminate coefficients a, b, c being real.

The formulas of displacements obtained at this solution have been complicated by the relations between the coefficients of the complex arguments. Thus the direct way of the solution of Lamé's equations of equilibrium has proved to be inconvenient for technical application.

The solution of Lamé's equations of equilibrium in indirect way in

conformance with the formulas of unit stresses (3) and (4) gives the following expressions of displacements

$$\frac{u}{\frac{\partial\vartheta}{\partial x}} = \frac{v}{\frac{\partial\vartheta}{\partial y}} = \frac{w}{\frac{\partial\vartheta}{\partial z}} = \frac{1}{2\mu} = \frac{1+\sigma}{E} \tag{10}$$

at the condition (5) and when $q = 0$.

Hence the rotations at every point of a solid are

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0, \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0,$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0,$$

i.e. this deformation is irrotational. In this particular case Lamé's equations of equilibrium become (see E. Sarrau. Notions sur la théorie de l'élasticité, Paris, 1889, paragraph 40, p. 38)

$$\frac{\partial\Delta}{\partial x} = 0, \qquad \frac{\partial\Delta}{\partial y} = 0, \qquad \frac{\partial\Delta}{\partial z} = 0;$$

they give the cubical dilatation

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \text{Const.} \tag{11}$$

The volume deformation of a body is characterized by the cubical dilatation Δ, which therefor is called by some Russian authors as a coefficient of volume deformation.

Having the expressions: of the normal unit stresses

$$X_x = \lambda\cdot\Delta + 2\mu\frac{\partial u}{\partial x}, \qquad Y_y = \lambda\cdot\Delta + 2\mu\frac{\partial v}{\partial y}, \qquad Z_z = \lambda\cdot\Delta + 2\mu\frac{\partial w}{\partial z}, \tag{12}$$

and the tangential unit stresses

$$\left.\begin{aligned} X_y = Y_x &= \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \\ Y_z = Z_y &= \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \\ Z_x = X_z &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \end{aligned}\right\} \tag{13}$$

and the relations between the coefficients of elasticity

$$\frac{\lambda}{\sigma} = \frac{2\mu}{1-2\sigma} = \frac{\lambda+2\mu}{1-\sigma} = \frac{3\lambda+2\mu}{1+\sigma} = 2(\lambda+\mu) = \frac{E}{(1+\sigma)(1-2\sigma)}, \quad (14)$$

it will not be difficult to verify that the substitution of the values (10) into (12) and (13) will give the formulas (3) and (4).

The more general solution of Lamé's equations of equilibrium in indirect way are B. G. Galerkin's formulas published in the above Russian edition in 1931. Bearing in mind the adopted denotation (6) and the condition (9) we may write this solution corresponding to the stresses (7) and (8) as follows

$$\left.\begin{aligned} u &= \frac{1+\sigma}{E}\left[-\frac{\partial\theta}{\partial x} + 2(1-\sigma)\nabla^2\psi_1\right] + Cy - Bz + D_1, \\ v &= \frac{1+\sigma}{E}\left[-\frac{\partial\theta}{\partial y} + 2(1-\sigma)\nabla^2\psi_2\right] + Az - Cx + D_2, \\ w &= \frac{1+\sigma}{E}\left[-\frac{\partial\theta}{\partial z} + 2(1-\sigma)\nabla^2\psi_3\right] + Bx - Ay + D_3, \end{aligned}\right\} \quad (15)$$

where A, B, C, D_i, $(i = 1; 2; 3)$ are arbitrary constants.

Three solutions of Lamé's equations of equilibrium obtained in indirect way were published by the author in the above collection of his works. The first solution was given in 1923 in the third article (in French with a Russian summary) in a symmetrical form by means of three arbitrary functions, each of which must satisfy the ultra-harmonic equation

$$\nabla\nabla^2\nabla^2 f(x, y, z) = 0, \quad (16)$$

where

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \quad (17)$$

In 1930 in the eighth article there were given two other solutions also in a symmetrical form as a result of generalization of Hertz and Boussinesq's unsymmetrical solutions.

As the first solution of the author proved to be too complicated, while the two last ones could be obtained from the reproduced below P. F. Papkovitch's solution, they have not been inserted here.

Analysing some solutions of Lamé's equations of equilibrium in a vectorial form in "Bulletin de l'Académie des sciences de l'URSS", 1932, No. 10, P. F. Papkovitch gave a similar general solution but in a simpler

form also in indirect way by means of the four harmonic functions φ_1, φ_2, φ_3 and φ_0. Introducing for brevity the notation

$$F = x\varphi_1 + y\varphi_2 + z\varphi_3 + \varphi_0, \tag{18}$$

this solution can be represented as follows

$$\left.\begin{aligned} u &= \varphi_1 - \frac{1}{4(1-\sigma)}\frac{\partial F}{\partial x}, \\ v &= \varphi_2 - \frac{1}{4(1-\sigma)}\frac{\partial F}{\partial y}, \\ w &= \varphi_3 - \frac{1}{4(1-\sigma)}\frac{\partial F}{\partial z}, \end{aligned}\right\} \tag{19}$$

where

$$\nabla^2\varphi_i(x, y, z) = 0, \ (i = 1; 2; 3; 0). \tag{20}$$

Hence

$$\Delta = \frac{1-2\sigma}{2(1-\sigma)}\left(\frac{\partial\varphi_1}{\partial x} + \frac{\partial\varphi_2}{\partial y} + \frac{\partial\varphi_3}{\partial z}\right) \tag{21}$$

is an expression for determining the cubical dilatation.

HARBIN.

THE GAUSS-CODAZZI-RICCI EQUATIONS IN RIEMANNIAN MANIFOLDS

By Abraham Schwartz

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I. Introduction

T. Y. Thomas has demonstrated the interesting theorem which says that for the case of a Riemannian manifold of $n - 1$ dimensions, V_{n-1},[1] imbedded in a Euclidean space of n dimensions, E_n, the Codazzi equation follows from the Gauss equation when the rank of the second fundamental form is ≥ 4.[2]

In this paper, we shall be concerned with the more general case of a V_m imbedded in a V_n, and we shall study the relationships between the Gauss, Codazzi, and Ricci equations in that case. The theorem of Thomas will be shown to be a special case of one of our theorems, although the method we shall use is quite different from that of Thomas, being more geometrical in nature.

We shall work in the notation used by J. A. Schouten and D. J. Struik in their book "Einfuhrung in die Neueren Methoden der Differentialgeometrie,"[3] and for the most part, we shall not presuppose anything not discussed in that book.

A. **The Normal Spaces for the $V_m \subset V_n$.** The normal spaces are defined by Schouten and Struik[4] in essentially the same way that Walther Mayer[5] defines them. Briefly, the surface, the V_m, is defined by the equations

$$(1.1)\quad \xi^\kappa = \xi^\kappa(\eta^{r_0}),\ \kappa, \lambda, \nu, \cdots = 1, \cdots, n;\ p_0, q_0, r_0, \cdots = 1, \cdots, m.$$

Let C be an arbitrary, non-isotropic curve of the surface passing through a given point P, and let its tangent vector be i^κ, its arc length z. The

[1] For the definition of a Riemannian manifold see I, pg. 84, 85.

[2] T. Y. Thomas, Acta Mathematica, vol. 67 (1935), pgs. 189 ff.

[3] J. A. Schouten und D. J. Struik, "Einfuhrung in die Neueren Methoden der Differentialgeometrie," Noordhoff, Groningen, 1935, vol. 1; 1938, vol. 2. Hereafter we shall refer to vol. 1 by I and to vol. 2 by II.

[4] II, pgs. 79–85, 116, 117.

[5] Walther Mayer, Trans. Am. Math. Soc., vol. 38 (1935), pgs. 267–309.

first curvature vectors, $\delta_x i^\kappa$, of all such curves C together with their tangent vectors, i^κ, define the first tangent space. The vectors i^κ, $\delta_x i^\kappa$, $\delta_x^2 i^\kappa = \delta_x \delta_x i^\kappa$ for all these curves C define the second tangent space, and similarly, the vectors i^κ, $\delta_x i^\kappa$, $\cdots$, $\delta_x^x i^\kappa$ for the curves C define the x^{th} tangent space. Because the surrounding space, the V_n, is of finitely many dimensions, there can only be finitely many distinct tangent spaces, say k. In other words, the vectors $\delta_x^M i^\kappa$, $M > k$, for the curves C are all dependent on the vectors i^κ, $\delta_x i^\kappa$, $\cdots$, $\delta_x^k i^\kappa$ for some k.

The V_m itself is called the 0^{th} normal space, and that part of the first tangent space normal to the 0^{th} normal space is called the first normal space. The second normal space is defined as that part of the second tangent space which is normal to the 0^{th} and 1^{st} normal spaces, and similarly, the x^{th} normal space is defined as that part of the x^{th} tangent space which is normal to the 0^{th}, $\cdots$, $(x-1)^{\text{st}}$ normal spaces. Each normal space is thus normal to every other normal space, and there are k normal spaces, all imbedded within the surrounding V_n.

The x^{th} normal space, $x = 0, 1, \cdots, k$, is of $\underset{x}{m}$ dimensions, $\underset{x}{m} \leq \frac{(\underset{0}{m})^x(\underset{0}{m}+1)}{2}$, $\underset{0}{m} = m$, $\sum_{x=0}^{k} \underset{x}{m} \leq n$, and therefore in it we can define a system of $\underset{x}{m}$ mutually orthogonal unit vectors. We agree to use Greek letters for indices referring to the V_n and the Latin letters p, q, r, s, t, u with subscripts x for indices referring to the x^{th} normal space, and thus, since the unit vectors which span the x^{th} normal space lie in the surrounding space, the V_n, as well as in the x^{th} normal space, we may describe them in any of the following ways:

$$\underset{p_x}{i^\kappa}, \; i_\lambda, \; \underset{p_x}{i^{q_x}}, \; \underset{p_x}{i_{r_x}}. \tag{1.2}$$

As almost always in this notation, the indices placed beside the kernel letter i ($\kappa, \lambda, q_x, r_x$ in this case) are the contravariant or covariant indices of the vectors; while the letters placed directly above or below the kernel letter (p_x in this case) are merely labels. For instance, $\underset{4}{i^\kappa}$ would be a contravariant expression in terms of V_n coordinates for the unit vector which was designated as the fourth of the set of $\underset{x}{m}$.

The affinor $\underset{0}{B^\kappa_{r_0}}$ is defined by the relationship

$$\underset{0}{B^\kappa_{r_0}} = \frac{\partial \xi^\kappa}{\partial \eta^{r_0}} = \partial_{r_0} \xi^\kappa = \xi^\kappa_{;r_0}.^6 \tag{1.3}$$

[6] I, Section 9; II, pgs. 117, 118.

It has the property

$$\overset{0}{B}{}^{x}_{r_0} W_x = \begin{cases} 0 & \text{if } W^x \text{ is a vector normal to the } V_m \\ W_{r_0} & \text{if } W^x \text{ is a vector lying in the } V_m, \end{cases} \tag{1.4}$$

and it may be expressed in terms of the unit vectors $\overset{p_0}{i}{}^{x}$ which span the 0^{th} normal space, the V_m.

$$\overset{0}{B}{}^{x}_{r_0} = \overset{p_0}{i}{}^{x} \overset{p_0}{i}_{r_0}. \tag{1.5}$$

Affinors $\overset{s}{B}{}^{x}_{r_s}$ which play analagous roles in the various other normal spaces are also defined.[6] They can be expressed in terms of the unit vectors $\overset{p_s}{i}{}^{x}$ which span the s^{th} normal space,

$$\overset{s}{B}{}^{x}_{r_s} = \overset{p_s}{i}{}^{x} \overset{p_s}{i}_{r_s}, \tag{1.6}$$

and they therefore have the property

$$\overset{s}{B}{}^{x}_{r_s} W_x = \begin{cases} 0 & \text{if } W^x \text{ is a vector normal to the } s^{\text{th}} \text{ normal space} \\ W_{r_s} & \text{if } W^x \text{ is a vector lying in the } s^{\text{th}} \text{ normal space.} \end{cases} \tag{1.7}$$

There exists also a fundamental tensor for the s^{th} normal space which can be used for the raising and lowering of indices referring to this s^{th} normal space. It is related to the fundamental tensor of the V_n, $a_{\lambda x}$, in the following way:

$$\overset{s}{a}_{p_s q_s} = \overset{s}{B}{}^{\lambda x}_{p_s q_s} a_{\lambda x}. \tag{1.8}$$

B. **The Frenet Equations.** The Frenet equations for a $V_m \subset V_n$ are arrived at in a way suggested by the Frenet equations of the curve theory of ordinary differential geometry.[7] They can be written in either of the following convenient forms:

$$\begin{gathered} D_{r_0} \overset{s}{B}{}^{x}_{p_s} = -\overset{s}{H}{}^{\cdot\cdot x}_{r_0 \cdot p_s} + \overset{s+1}{H}{}^{\cdot\cdot x}_{r_0 p_s}, \qquad x = 0, \cdots, k, \\ \overset{0}{H}{}^{\cdot\cdot x}_{r_0 \cdot p_0} = 0, \qquad \overset{k+1}{H}{}^{\cdot\cdot x}_{r_0 p_k} = 0. \end{gathered} \tag{1.9}$$

$$\begin{gathered} D_{r_0} \overset{s}{B}{}^{q_s}_{x} = -\overset{s}{H}{}^{\cdot\cdot q_s}_{r_0 x} + \overset{s+1}{H}{}^{q_s}_{r_0 \cdot x}, \qquad x = 0, \cdots, k, \\ \overset{0}{H}{}^{\cdot\cdot q_0}_{r_0 x} = 0, \qquad \overset{k+1}{H}{}^{q_k}_{r_0 \cdot x} = 0. \end{gathered} \tag{1.10}$$

[7] II, pgs. 117–120.

In these equations we have used what Schouten and Struik call the "D-notation" to indicate our derivatives.[8] To describe it briefly for our purposes, we note that if X^{r_0} is a vector of the V_m, $D_{s_0}X^{r_0}$ is the covariant derivative of X^{r_0} with respect to the V_m.

$$D_{s_0}X^{r_0} = \partial_{s_0}X^{r_0} + \left\{ \begin{matrix} r_0 \\ s_0\, t_0 \end{matrix} \right\} X^{t_0} = X^{r_0}_{;s_0} = \nabla'_{s_0}X^{r_0}. \tag{1.11}$$

If Y^κ is a vector of the V_n, $D_\lambda Y^\kappa$ is the covariant derivative of Y^κ with respect to the V_n.

$$D_\lambda Y^\kappa = \partial_\lambda Y^\kappa + \left\{ \begin{matrix} \kappa \\ \lambda\nu \end{matrix} \right\} Y^\nu = Y^\kappa_{;\lambda} = \nabla_\lambda Y^\kappa \tag{1.12}$$

$D_{r_0}Y^\kappa$ is the "V_m-projection" of $D_\lambda Y^\kappa$,

$$D_{r_0}Y^\kappa = \overset{0}{B}{}^\lambda_{r_0} D_\lambda Y^\kappa = Y^\kappa_{;r_0} = \overset{0}{B}{}^\lambda_{r_0} \nabla_\lambda Y^\kappa. \tag{1.13}$$

In this last equation we have indicated a comparison between the D-notation and the "generalized covariant differentiation."[9] For purposes of further comparison, if $A^\kappa_{\cdot\lambda r_0}$ is an affinor lying with its first two indices in the V_n and with the third on the V_m, then in the generalized covariant differentiation,

$$A^\kappa_{\cdot\lambda r_0;s_0} = \frac{\partial A^\kappa_{\cdot\lambda r_0}}{\partial \eta^{s_0}} + \left\{ \begin{matrix} \kappa \\ \nu\tau \end{matrix} \right\} A^\nu_{\cdot\lambda r_0} \xi^\tau_{;s_0} - \left\{ \begin{matrix} \nu \\ \lambda\tau \end{matrix} \right\} A^\kappa_{\cdot\nu r_0} \xi^\tau_{;s_0} - \left\{ \begin{matrix} t_0 \\ r_0\, s_0 \end{matrix} \right\} A^\kappa_{\cdot\lambda t_0}, \tag{1.14}$$

and in the D-notation,

$$\begin{aligned} D_{s_0}A^\kappa_{\cdot\lambda r_0} &= \overset{0}{B}{}^\tau_{s_0} D_\tau A^\kappa_{\cdot\lambda r_0} = \overset{0}{B}{}^\tau_{s_0} \frac{\partial A^\kappa_{\cdot\lambda r_0}}{\partial \xi^\tau} \\ &+ \left\{ \begin{matrix} \kappa \\ \nu\tau \end{matrix} \right\} A^\nu_{\cdot\lambda r_0} \overset{0}{B}{}^\tau_{s_0} - \left\{ \begin{matrix} \nu \\ \lambda\tau \end{matrix} \right\} A^\kappa_{\cdot\nu r_0} \overset{0}{B}{}^\tau_{s_0} - \left\{ \begin{matrix} t_0 \\ r_0\, s_0 \end{matrix} \right\} A^\kappa_{\cdot\lambda t_0}. \end{aligned} \tag{1.15}$$

We note here that $\overset{0}{B}{}^\tau_{s_0} = \dfrac{\partial \xi^\tau}{\partial \eta^{s_0}} = \xi^\tau_{;s_0}$, (1.3).

The affinors $\overset{x}{H}{}^{\cdot\cdot\lambda}_{r_0\kappa}$ are defined by means of these Frenet equations. $\overset{x}{H}{}^{\cdot\cdot\lambda}_{r_0\kappa}$ lies in the x^{th} normal space with respect to its third index, in the $(x-1)^{\text{st}}$ normal space with respect to its second index, and on the V_m

[8] This powerful symbolism is explained in detail in I, Sec. 9 and in II, pg. 117.

[9] See C. E. Weatherburn, "Introduction to Riemannian Geometry and the Tensor Calculus," University Press, Cambridge, England, 1938.

with respect to its first index.[10] Further, it is of rank m with respect to its third index.[7] We shall be very much concerned with these affinors in the work that follows.

Essentially, the Frenet equations tell us that the covariant derivative of a vector lying in the x^{th} normal space lies in the $(x-1)^{\text{st}}$, x^{th}, $(x+1)^{\text{st}}$ normal spaces and in no others.[11]

C. **The Gauss-Codazzi-Ricci Equations.** The Gauss-Codazzi-Ricci equations follow now formally from the Frenet equations by differentiation, alternation on some of the indices, and projection in the various normal spaces. We state these equations in a form equivalent to that appearing in the book of Schouten and Struik.[12]

$$
\begin{aligned}
&\overset{x}{N}\ \overset{0}{B}{}^{\nu\mu}_{s_0r_0}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x}{B}{}^{\kappa}_{p_x}K_{\nu\mu\lambda\kappa} = \overset{x}{K}_{s_0r_0q_xp_x} - 2\overset{x}{H}_{[s_0|}{}^{\cdot\,\cdot\,p_{x-1}}{}_{|p_x|}\overset{x}{H}_{r_0]p_{x-1}q_x}\\
&\qquad\qquad - 2H_{[s_0|}{}^{\cdot\,\cdot\,p_x}{}_{|p_x|}\overset{x+1}{H}{}^{p_{x+1}}_{r_0]q_xp_{x+1}}, \qquad x = 0, 1, \cdots, k,\\
&\overset{x}{N}_{x-1}\ \overset{0}{B}{}^{\nu\mu}_{s_0r_0}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x-1}{B}{}^{\kappa}_{p_{x-1}}K_{\nu\mu\lambda\kappa} = -2D_{[s_0}\overset{x}{H}_{r_0]p_{x-1}q_x}, \qquad x = 1, \cdots, k,\\
&\overset{x}{N}_{x-2}\ \overset{0}{B}{}^{\nu\mu}_{s_0r_0}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x-2}{B}{}^{\kappa}_{p_{x-2}}K_{\nu\mu\lambda\kappa}\\
&\qquad\qquad = 2H_{[s_0|}{}^{\cdot\,\cdot\,p_{x-1}}{}_{|p_{x-2}|}\overset{x-1}{H}{}^{\cdot\,\cdot\,}_{r_0]p_{x-1}q_x}, \qquad x = 2, \cdots, k,\\
&\overset{x}{N}_{x-g}\ \overset{0}{B}{}^{\nu\mu}_{s_0r_0}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x-g}{B}{}^{\kappa}_{p_{x-g}}K_{\nu\mu\lambda\kappa} = 0, \qquad g \geq 3, \qquad x = g, \cdots, k.
\end{aligned}
\tag{1.16}
$$

The affinors $K_{\nu\mu\lambda\kappa}$ are the Riemann-Christoffel affinors for the V_n; while the affinors $\overset{x}{K}_{s_0r_0q_xp_x}$ are defined in a similar way, namely by the equation

$$
D_{[s_0}D_{r_0]}w_{q_x} = -\tfrac{1}{2}\overset{x}{K}_{s_0r_0q_x}{}^{\cdot\,\cdot\,p_x}w_{p_x}, \tag{1.17}
$$

where w_{q_x} is an arbitrary vector of the x^{th} normal space.

[10] See equations (3.6) and (3.7) below.

[11] W. Mayer arrives at a form of the Frenet equations which also illustrates clearly this point. See pages 282-293 of the paper mentioned in footnote 5.

[12] II, pgs. 121–125.

The square brackets, [ ], indicate alternation on the indices included between them.[13] For instance,

$$(1.18)\quad x_{[r_0 s_0]} = \frac{1}{\underline{|2}}\,(x_{r_0 s_0} - x_{s_0 r_0})$$

$$(1.19)\quad \underline{|2}\, D_{[s_0} \overset{x}{H}_{r_0]p_{x-1}q_x} = D_{s_0} \overset{x}{H}_{r_0 p_{x-1} q_x} - D_{r_0} \overset{x}{H}_{s_0 p_{x-1} q_x}$$

$$(1.20)\quad 2\overset{x}{H}_{[s_0 \vdots}{}^{p_{x-1}}{}_{|p_x|} \overset{x}{H}_{r_0]p_{x-1}q_x} = \overset{x}{H}_{s_0 \vdots}{}^{p_{x-1}}{}_{p_x} \overset{x}{H}_{r_0 p_{x-1} q_x} - \overset{x}{H}_{r_0 \vdots}{}^{p_{x-1}}{}_{p_x} \overset{x}{H}_{s_0 p_{x-1} q_x}.$$

The symbols $\overset{y}{\underset{x}{N}}$ are only labels for the various equations. The upper label indicates the normal space with which the transvection on the third index of the affinor K is concerned; the lower label indicates the normal space with which the transvection on the fourth index of the affinor K is concerned. The equations $\overset{x}{\underset{x}{N}}$ (upper label = lower label) are usually called the Gauss equations; the equations $\overset{x}{\underset{x-1}{N}}$ (upper label = lower label + 1) are usually called the Codazzi equations; and the equations $\overset{x}{\underset{x-2}{N}}$ (upper label = lower label + 2) are usually called the Ricci equations.

To consider these Gauss-Codazzi-Ricci equations in a case more familiar perhaps, let us consider the case of a $V_{n-1} \subset R_n$. Then there can only be one normal space, the first, and it is of one dimension at most, $m \leq 1$. Further, the Riemann-Christoffel affinor for the R_n is zero, and the Gauss-Codazzi-Ricci equations reduce to

$$(1.21)\quad \overset{0}{\underset{0}{N}}\quad \overset{0}{K}_{s_0 r_0 q_0 p_0} = 2\overset{1}{H}_{[s_0|p_0|} \overset{1}{H}_{r_0]q_0 p_1} = \overset{1}{H}_{s_0 p_0} \overset{1}{H}_{r_0 q_0 p_1} - \overset{1}{H}_{r_0 p_0} \overset{1}{H}_{s_0 q_0 p_1},$$

$$(1.22)\quad \overset{1}{\underset{0}{N}}\quad 2D_{[s_0} \overset{1}{H}_{r_0]p_0 q_1} = D_{s_0} \overset{1}{H}_{r_0 p_0 q_1} - D_{r_0} \overset{1}{H}_{s_0 p_0 q_1} = 0.$$

If $m = 1$, we can write

$$(1.23)\quad \overset{1}{H}_{r_0 q_0}{}^{p_1} = h_{r_0 q_0} \overset{p_1}{\underset{1}{i}},$$

where $\overset{p_1}{\underset{1}{i}}$ is the vector of the V_n normal to the V_m and where $h_{r_0 q_0}$ is

[13] This symbolism is explained in detail in I, pgs. 14, 15.

what we usually call the second fundamental form.[14] (1.21) reduces directly to

$$\overset{0}{N}_{0}\ \overset{0}{K}_{s_0 r_0 q_0 p_0} = h_{s_0 p_0} h_{r_0 q_0} - h_{r_0 p_0} h_{s_0 q_0}, \tag{1.24}$$

[15]

and the i^{th} component of (1.22) becomes

$$\overset{1}{N}_{0}\ D_{s_0} h_{r_0 p_0} - D_{r_0} h_{s_0 p_0} = 0. \tag{1.25}$$

[15]

II. The Conditions Imposed on the Curvature Affinors $\overset{x}{H}{}^{\cdots q_x}_{r_0 p_{x-1}}$

A. **The Definition of Condition 0 for $\overset{x}{H}$.** In our work we shall very often impose certain conditions on the affinors $\overset{x}{H}$. In order not to break the continuity of the proofs of our theorems, we shall consider these conditions now.

We shall require first that there exist $\underset{x}{\alpha}$ independent vectors of the V_m, $\overset{x}{u}{}^{r_0}_{A}$, $A = 1, \cdots, \underset{x}{\alpha}$, such that

$$\overset{x}{u}{}^{r_0}_{A}\ \overset{x}{H}{}^{\cdots q_x}_{r_0 p_{x-1}} = 0. \tag{2.1}$$

Again we have followed the notation conventions of Schouten and Struik. Since the vectors we are describing, $\overset{x}{u}{}^{r_0}_{A}$, are vectors of the V_m, the 0^{th} normal space, we can use letters p_0, r_0, s_0 etc. for the running indices. The upper label x indicates that property (2.1) holds if the x^{th} curvature affinor, $\overset{x}{H}$, be considered, and the lower label A indicates that $\overset{x}{u}{}^{r_0}_{A}$ is the A^{th} such vector. The choice of the letter u as kernel index is purely arbitrary. Similar explanations hold for the notation used in describing the vectors $\overset{x}{v}{}^{r_0}_{B}$, $\overset{x}{z}{}^{p_{x-1}}_{B}$, $\overset{x}{y}{}^{p_{x-1}}_{C}$ below.

Then we shall ask that there exist $\underset{x}{\beta} = \underset{0}{m} - \underset{x}{\alpha}$ independent vectors of

[14] See (6.6) and (6.7) below.

[15] These equations are those of L. P. Eisenhart for this case. See "Riemannian Geometry," Princeton University Press, 1926, pg. 149. For the case of a $V_m \subset V_n$ Eisenhart arrives at Gauss-Codazzi-Ricci equations which do not deal with the various normal spaces separately. See pg. 162 of the above mentioned book and also II, pg. 130.

the V_m, $\overset{x}{v}{}^{r_0}_{B}$, $B = 1, \cdots, \beta$, and β independent vectors of the $(x-1)^{st}$ normal space, $\overset{x}{z}{}^{p_{x-1}}_{B}$, such that

$$\overset{x}{v}{}^{r_0}_{B_1}\overset{x}{z}{}^{p_{x-1}}_{B_2}\overset{x}{H}{}^{\cdots q_x}_{r_0 p_{x-1}} \begin{cases} = 0, & B_1 \neq B_2 \\ \neq 0, & B_1 = B_2. \end{cases} \tag{2.2}$$

Finally, we shall ask that there exist $\gamma = m - \beta$ independent vectors of the $(x-1)^{st}$ normal space, $\overset{x}{y}{}^{p_{x-1}}_{C}$, $C = 1, \cdots, \gamma$, such that

$$\overset{x}{y}{}^{p_{x-1}}_{C}\overset{x}{H}{}^{\cdots q_x}_{r_0 p_{x-1}} = 0. \tag{2.3}$$

If these three conditions are satisfied, we shall say that condition 0 for $\overset{x}{H}$ is satisfied. The reason for this name will appear presently.

1. *A Necessary Consequence of Condition* 0 *for* $\overset{x}{H}$. The affinor $\overset{x}{H}{}^{\cdots q_x}_{r_0 p_{x-1}}$ is of rank m with respect to the index q_x.[16] This means that if we take all possible transvections on the indices r_0 and p_{x-1} we shall arrive at exactly m independent vectors.[17]

Now let X^{r_0} be an arbitrary vector of the V_m. Because the vectors $\overset{x}{u}{}^{r_0}_{A}$, $\overset{x}{v}{}^{r_0}_{B}$ constitute a set of m independent vectors of the V_m, we can say that

$$X^{r_0} = \overset{A}{X}\overset{x}{u}{}^{r_0}_{A} + \overset{B}{\bar{X}}\overset{x}{v}{}^{r_0}_{B} \tag{2.4}$$

where $\overset{A}{X}$ and $\overset{B}{\bar{X}}$ are scalars. Similarly, if $Y^{p_{x-1}}$ is an arbitrary vector of the $(x-1)^{st}$ normal space, we can use the fact that $\overset{x}{z}{}^{p_{x-1}}_{B}$, $\overset{x}{y}{}^{p_{x-1}}_{C}$ constitute a set of m independent vectors in the $(x-1)^{st}$ normal space, and we can say that

$$Y^{p_{x-1}} = \overset{B}{Y}\overset{x}{z}{}^{p_{x-1}}_{B} + \overset{C}{\bar{Y}}\overset{x}{y}{}^{p_{x-1}}_{C} \tag{2.5}$$

where $\overset{B}{Y}$ and $\overset{C}{\bar{Y}}$ are scalars.

[16] II, pg. 120.
[17] I, pg. 19.

An arbitrary transvection on the indices r_0, p_{x-1} of the affinor $\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}$ can then be considered as follows when one assumes that condition 0 for $\overset{x}{H}$ is satisfied:

$$X^{r_0}Y^{p_{x-1}}\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x} = \left(\overset{A\,x}{X}u^{r_0} + \overset{B_1 x}{\underset{B_1}{X}}v^{r_0}\right)\left(\overset{B_2 x}{\underset{B_2}{Y}}z^{p_{x-1}} + \overset{C\,x}{\underset{C}{Y}}y^{p_{x-1}}\right)\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}$$

(2.6)

$$= \overset{B_1 B_2 x}{\underset{B_1 B_2}{X\,Y}}v^{r_0}z^{p_{x-1}}\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}.$$

Here we have used (2.1) and (2.3). If, in addition, we remember that there must be $\overset{x}{m}$ independent vectors of the form $X^{r_0}Y^{p_{x-1}}\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}$ because the rank of $\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}$ with respect to q_x is just $\overset{x}{m}$, then (2.2) and (2.6) tell us that there are exactly $\overset{x}{m}$ independent vectors in the set of $\overset{x}{\beta}$ vectors $\underset{B}{\overset{x}{v}}^{r_0}\underset{B}{\overset{x}{z}}^{p_{x-1}}\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}$. We can state this result as

LEMMA II. If condition 0 for $\overset{x}{H}$ is satisfied, then there are exactly $\overset{x}{m}$ independent vectors among $\underset{B}{\overset{x}{v}}^{r_0}\underset{B}{\overset{x}{z}}^{p_{x-1}}\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}$.

B. **The Definition of Condition A for $\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}$.** In order to be able to deal with certain sets of linear equations that we shall consider later, we shall have to require of the $\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}$ that in every set of $\overset{x}{\beta} - A$ vectors selected from the set of $\overset{x}{\beta}$ vectors $\underset{B}{\overset{x}{v}}^{r_0}\underset{B}{\overset{x}{z}}^{p_{x-1}}\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}$ there shall be $\overset{x}{m}$ independent vectors. If both condition 0 for $\overset{x}{H}$ and this additional condition are satisfied, we shall say that condition A for $\overset{x}{H}$ is satisfied.

We might note here that condition A for $\overset{x}{H}$ can only be satisfied if $\overset{x}{\beta} \geq \overset{x}{m} + A$, and that if condition A_1 for $\overset{x}{H}$ is satisfied then condition A_2 for $\overset{x}{H}$ is also satisfied when $A_2 < A_1$.

If $\overset{x}{m} = 1$, the condition that $\overset{x}{\beta} \geq \overset{x}{m} + A = 1 + A$ is not only a necessary condition that condition A for $\overset{x}{H}$ shall be satisfied but also

a sufficient condition. This follows from the facts that there are no null vectors in the set $\overset{s}{B}\overset{s}{v}^{r_0}\overset{s}{z}^{p_{s-1}}\overset{s}{H}_{r_0 p_{s-1}\cdot\cdot\cdot q_s}$,[18] and that there is always one independent vector in a set of one or more non-null vectors. We can state this result as

LEMMA II.2 If $\underset{s}{m} = 1$, then when condition A for $\overset{s}{H}$ is satisfied, $\underset{s}{\beta} \geq A + 1$, and when $\underset{s}{\beta} \geq A + 1$, condition A for $\overset{s}{H}$ is satisfied.

III. Theorems Arrived at by Working with the Codazzi Equation $\overset{s}{N}_{s-1}$

A. $\overset{s}{N}_{s-1}, \overset{s-1}{N}_{s-1}, \overset{s}{N}_{s-2}, \overset{s+1}{N}_{s-1} \to \overset{s}{N}$.

In this section we shall show that the Gauss equation $\overset{s}{N}_{s}$ depends on some of the other Gauss-Codazzi-Ricci equations under certain conditions.

We can write the equation $\overset{s}{N}_{s-1}$ in the following form:

$$(3.1) \equiv (1.16)\quad \overset{0}{B}{}^{\nu\mu}_{s_0 r_0}\overset{s}{B}{}^{\lambda}_{q_s}\overset{s-1}{B}{}^{\kappa}_{p_{s-1}} K_{\nu\mu\lambda\kappa} = -2D_{[s_0}\overset{s}{H}_{r_0]p_{s-1}q_s}, \quad s = 1, \cdots, k.$$

Now we differentiate both sides of this equation, and we have

$$\begin{aligned}(3.2)\quad &(D_{u_0}\overset{0}{B}{}^{\nu}_{s_0})\overset{0}{B}{}^{\mu}_{r_0}\overset{s}{B}{}^{\lambda}_{q_s}\overset{s-1}{B}{}^{\kappa}_{p_{s-1}}K_{\nu\mu\lambda\kappa} + (D_{u_0}\overset{0}{B}{}^{\mu}_{r_0})\overset{0}{B}{}^{\nu}_{s_0}\overset{s}{B}{}^{\lambda}_{q_s}\overset{s-1}{B}{}^{\kappa}_{p_{s-1}}K_{\nu\mu\lambda\kappa}\\ &+ (D_{u_0}\overset{s}{B}{}^{\lambda}_{q_s})\overset{0}{B}{}^{\nu\mu}_{s_0 r_0}\overset{s-1}{B}{}^{\kappa}_{p_{s-1}}K_{\nu\mu\lambda\kappa} + (D_{u_0}\overset{s-1}{B}{}^{\kappa}_{p_{s-1}})\overset{0}{B}{}^{\nu\mu}_{s_0 r_0}\overset{s}{B}{}^{\lambda}_{q_s}K_{\nu\mu\lambda\kappa}\\ &+ \overset{0}{B}{}^{\nu\mu}_{s_0 r_0}\overset{s}{B}{}^{\lambda}_{q_s}\overset{s-1}{B}{}^{\kappa}_{p_{s-1}}D_{u_0}K_{\nu\mu\lambda\kappa} = -2D_{u_0}D_{[s_0}\overset{s}{H}_{r_0]p_{s-1}q_s}.\end{aligned}$$

We can make use of the Frenet equations to change the form of the left hand member of (3.2), and we use (1.9) and (1.10) for that purpose.

$$\begin{aligned}(3.3)\quad &\overset{1}{H}_{u_0 s_0}\overset{0}{B}{}^{\nu}_{r_0}\overset{s}{B}{}^{\lambda}_{q_s}\overset{s-1}{B}{}^{\kappa}_{p_{s-1}}K_{\nu\mu\lambda\kappa} + \overset{1}{H}_{u_0 r_0}\overset{0}{B}{}^{\mu}_{s_0}\overset{s}{B}{}^{\lambda}_{q_s}\overset{s-1}{B}{}^{\kappa}_{p_{s-1}}K_{\nu\mu\lambda\kappa}\\ &+ (-\overset{s}{H}_{u_0 q_s}{}^{\lambda} + \overset{s+1}{H}_{u_0 q_s}{}^{\lambda})\overset{0}{B}{}^{\nu\mu}_{s_0 r_0}\overset{s-1}{B}{}^{\kappa}_{p_{s-1}}K_{\nu\mu\lambda\kappa}\\ &+ (-\overset{s-1}{H}_{u_0 p_{s-1}}{}^{\kappa} + \overset{s}{H}_{u_0 p_{s-1}}{}^{\kappa})\overset{0}{B}{}^{\nu\mu}_{s_0 r_0}\overset{s}{B}{}^{\lambda}_{q_s}K_{\nu\mu\lambda\kappa}\\ &+ \overset{0}{B}{}^{\nu\mu}_{s_0 r_0}\overset{s}{B}{}^{\lambda}_{q_s}\overset{s-1}{B}{}^{\kappa}_{p_{s-1}}D_{u_0}K_{\nu\mu\lambda\kappa} = -2D_{u_0}D_{[s_0}\overset{s}{H}_{r_0]p_{s-1}q_s}.\end{aligned}$$

[18] See (2.2).

If we alternate on the indices u_0, s_0, r_0,[19] we have

$$
\begin{aligned}
&\overset{1}{H}_{[u_0 s_0 \cdot}{}^{\cdot}{}^{\cdot}{}^{\nu} \overset{0}{B}{}^{\mu}_{r_0]} \overset{x}{B}{}^{\lambda}_{q_x} \overset{x-1}{B}{}^{\kappa}_{p_{x-1}} K_{\nu\mu\lambda\kappa} + \overset{1}{H}_{[u_0 r_0 \cdot}{}^{\cdot}{}^{\cdot}{}^{\mu} \overset{0}{B}{}^{\nu}_{s_0]} \overset{x}{B}{}^{\lambda}_{q_x} \overset{x-1}{B}{}^{\kappa}_{p_{x-1}} K_{\nu\mu\lambda\kappa} \\
&+ (- \overset{x}{H}_{[u_0 \cdot}{}^{\cdot}{}^{\lambda}_{|q_x|} + \overset{x+1}{H}_{[u_0 |q_x|}{}^{\cdot}{}^{\cdot}{}^{\lambda}) \overset{0}{B}{}^{\nu\mu}_{s_0 r_0]} \overset{x-1}{B}{}^{\kappa}_{p_{x-1}} K_{\nu\mu\lambda\kappa} \\
&+ (- \overset{x-1}{H}_{[u_0 \cdot}{}^{\cdot}{}^{\kappa}_{|p_{x-1}|} + \overset{x}{H}_{[u_0 |p_{x-1}|}{}^{\cdot}{}^{\cdot}{}^{\kappa}) \overset{0}{B}{}^{\nu\mu}_{s_0 r_0]} \overset{x}{B}{}^{\lambda}_{q_x} K_{\nu\mu\lambda\kappa} \\
&+ \overset{0}{B}{}^{\nu\mu}_{[s_0 r_0} \overset{x}{B}{}^{\lambda}_{|q_x|} \overset{x-1}{B}{}^{\kappa}_{|p_{x-1}|} D_{u_0]} K_{\nu\mu\lambda\kappa} = -2 D_{[u_0} D_{s_0} \overset{x}{H}_{r_0] p_{x-1} q_x}.
\end{aligned} \tag{3.4}
$$

Here we know that

$$
\begin{aligned}
2 D_{[u_0} D_{s_0]} \overset{x}{H}_{r_0 p_{x-1} q_x} = &- \overset{0}{K}_{u_0 s_0 r_0}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{t_0} \overset{x}{H}_{t_0 p_{x-1} q_x} \\
&- \overset{x-1}{K}_{u_0 s_0 p_{x-1}}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{r_{x-1}} \overset{x}{H}_{r_0 r_{x-1} q_x} - \overset{x}{K}_{u_0 s_0 q_x}{}^{\cdot}{}^{\cdot}{}^{\cdot}{}^{r_x} \overset{x}{H}_{r_0 p_{x-1} r_x},
\end{aligned} \tag{3.5}
$$

[20]

and that

$$
\overset{x}{H}_{u_0 \cdot}{}^{\cdot}{}^{\lambda}_{q_x} = \overset{x-1}{B}{}^{\lambda}_{r_{x-1}} \overset{x}{H}_{u_0}{}^{\cdot}{}^{\cdot}{}^{r_{x-1}}_{\cdot\ q_x} \tag{3.6}
$$

$$
\overset{x}{H}_{u_0 p_{x-1}}{}^{\cdot}{}^{\cdot}{}^{\kappa} = \overset{x}{B}{}^{\kappa}_{r_x} \overset{x}{H}_{u_0 p_{x-1}}{}^{\cdot}{}^{\cdot}{}^{r_x}. \tag{3.7}
$$

[21]

Furthermore, the first two terms of the left hand side of (3.4) vanish because $\overset{1}{H}_{u_0 s_0}{}^{\cdot}{}^{\cdot}{}^{\nu}$ is symmetric with respect to its first two indices,[22] and the last term vanishes because of the Bianchi Identity for the Riemann-Christoffel affinor.[23] This can be seen if we rewrite that term as follows:

$$
\begin{aligned}
\overset{x}{B}{}^{\lambda}_{q_x} \overset{x-1}{B}{}^{\kappa}_{p_{x-1}} \overset{0}{B}{}^{\nu\mu}_{[s_0 r_0} D_{u_0]} K_{\nu\mu\lambda\kappa} &= \overset{x}{B}{}^{\lambda}_{q_x} \overset{x-1}{B}{}^{\kappa}_{p_{x-1}} \overset{0}{B}{}^{\nu\mu\sigma}_{[s_0 r_0 u_0]} D_\sigma K_{\nu\mu\lambda\kappa} \\
&= \overset{x}{B}{}^{\lambda}_{q_x} \overset{x-1}{B}{}^{\kappa}_{p_{x-1}} \overset{0}{B}{}^{\nu\mu\sigma}_{s_0 r_0 u_0} D_{[\sigma} K_{\nu\mu]\lambda\kappa} = 0.
\end{aligned} \tag{3.8}
$$

[19] The formal symbolism of alternation is explained in some detail in I, pgs. 14, 15. See also (1.18), (1.19), (1.20). Here, for instance,

$$
\begin{aligned}
-D_{[u_0} D_{s_0} \overset{x}{H}_{r_0] p_{x-1} q_x} = -\frac{1}{\boxed{3}} \{ &D_{u_0} D_{s_0} \overset{x}{H}_{r_0 p_{x-1} q_x} + D_{s_0} D_{r_0} \overset{x}{H}_{u_0 p_{x-1} q_x} \\
&+ D_{r_0} D_{u_0} \overset{x}{H}_{s_0 p_{x-1} q_x} - D_{u_0} D_{r_0} \overset{x}{H}_{s_0 p_{x-1} q_x} - D_{r_0} D_{s_0} \overset{x}{H}_{u_0 p_{x-1} q_x} \\
&- D_{s_0} D_{u_0} \overset{x}{H}_{r_0 p_{x-1} q_x} \}.
\end{aligned}
$$

[20] See (1.17).

[21] II, pg. 120.

[22] II, pg. 84.

[23] I, pg. 124.

The Second Identity for the Riemann-Christoffel affinor of the V_m also helps us to simplify (3.4).[24] It says that

$$\overset{0}{K}_{[u_0 s_0 r_0] p_0} = 0. \tag{3.9}$$

Now, using (3.5)–(3.9), we transpose all the right hand terms to the left hand side and rewrite (3.4). We have

$$\begin{aligned}
&-\overset{s}{H}_{[r_0 \cdots r_{s-1}|q_s|}(\overset{0}{B}{}^{\nu\mu}_{u_0 s_0]}\overset{s-1}{B}{}^{\lambda}_{r_{s-1}}\overset{s-1}{B}{}^{\varepsilon}_{p_{s-1}}K_{\nu\mu\lambda\varepsilon} + \overset{s-1}{K}_{u_0 s_0] p_{s-1} r_{s-1}}) \\
&\quad + \overset{s}{H}_{[r_0|p_{s-1}|\cdots r_s}(\overset{0}{B}{}^{\nu\mu}_{u_0 s_0]}\overset{s}{B}{}^{\lambda}_{q_s}\overset{s}{B}{}^{\varepsilon}_{r_s}K_{\nu\mu\lambda\varepsilon} - \overset{s}{K}_{u_0 s_0] q_s r_s}) \\
&\quad + \overset{s+1}{H}_{[r_0|q_s|\cdots r_{s+1}}\overset{0}{B}{}^{\nu\mu}_{u_0 s_0]}\overset{s+1}{B}{}^{\lambda}_{r_{s+1}}\overset{s-1}{B}{}^{\varepsilon}_{p_{s-1}}K_{\nu\mu\lambda\varepsilon} \\
&\quad - \overset{s-1}{H}_{[r_0 \cdots r_{s-2}|p_{s-1}|}\overset{0}{B}{}^{\nu\mu}_{u_0 s_0]}\overset{s}{B}{}^{\lambda}_{q_s}\overset{s-2}{B}{}^{\varepsilon}_{r_{s-2}}K_{\nu\mu\lambda\varepsilon} = 0.
\end{aligned} \tag{3.10}$$

Here the first term suggests the Gauss equation $\underset{s-1}{\overset{s-1}{N}}$, the second the Gauss equation $\underset{s}{\overset{s}{N}}$, the third the Ricci equation $\underset{s-1}{\overset{s+1}{N}}$, and the fourth the Ricci equation $\underset{s-2}{\overset{s}{N}}$. We might expect to arrive at these equations by adding to the terms of (3.10) the following group of terms composed of products of H's:

$$\begin{aligned}
&2\overset{s}{H}_{[r_0 \cdots r_{s-1}|q_s|}\overset{s-1}{H}_{u_0|p_{s-2}r_{s-1}|}\overset{s-1}{H}_{s_0]\cdots p_{s-1}} \\
&\quad + 2\overset{s}{H}_{[r_0 \cdots r_{s-1}|q_s|}\overset{s}{H}_{u_0|r_{s-1}r_s|}\overset{s}{H}_{s_0]p_{s-1}\cdots r_s} \\
&\quad + 2\overset{s}{H}_{[r_0|p_{s-1}|\cdots r_s}\overset{s}{H}_{u_0|r_{s-1}r_s|}\overset{s}{H}_{s_0]\cdots r_{s-1}q_s} \\
&\quad + 2\overset{s}{H}_{[r_0|p_{s-1}|\cdots r_s}\overset{s+1}{H}_{u_0|r_s|\cdots r_{s+1}}\overset{s+1}{H}_{s_0|q_s r_{s+1}} \\
&\quad - 2\overset{s+1}{H}_{[r_0|q_s|\cdots r_{s+1}}\overset{s}{H}_{u_0|p_{s-1}r_s|}\overset{s+1}{H}_{s_0]\cdots r_{s+1}} \\
&\quad + 2\overset{s-1}{H}_{[r_0 \cdots r_{s-2}|p_{s-1}|}\overset{s-1}{H}_{u_0|r_{s-2}r_{s-1}|}\overset{s}{H}_{s_0]\cdots r_{s-1}q_s}.
\end{aligned} \tag{3.11}$$

It is not hard to see that this group of terms vanishes identically. In fact, a slight rearrangement of the order of the factors in some of the terms shows quite quickly that the first and last, second and third, fourth and fifth terms cancel each other out.

[24] I, pg. 113.

Now if the terms of (3.11) be added to the left hand side of (3.10), we can write

$$
\begin{aligned}
&\overset{s}{H}_{[r_0\cdots r_{s-1}|q_s|}\Big(\overset{0}{B}{}^{\rho\mu}_{u_0\sigma_0]}\overset{s-1}{B}{}^{\lambda}_{p_{s-1}}\overset{s-1}{B}{}^{\kappa}_{r_{s-1}}K_{\rho\mu\lambda\kappa}-\overset{s-1}{K}_{u_0\sigma_0]p_{s-1}r_{s-1}}\\
&+2H_{u_0|p_{s-2}r_{s-1}|}\overset{s-1}{H}_{\sigma_0]}{}^{p_{s-2}}{}_{p_{s-1}}+2H_{u_0|r_{s-1}r_s|}\overset{s}{H}_{\sigma_0]p_{s-1}}{}^{r_s}\Big)\\
&+\overset{s}{H}_{[r_0|p_{s-1}|}{}^{r_s}\Big(\overset{0}{B}{}^{\rho\mu}_{u_0\sigma_0]}\overset{s}{B}{}^{\lambda}_{q_s}\overset{s}{B}{}^{\kappa}_{r_s}K_{\rho\mu\lambda\kappa}-\overset{s}{K}_{u_0\sigma_0]q_sr_s}\\
&+2H_{u_0|r_{s-1}r_s|}\overset{s}{H}_{\sigma_0]}{}^{r_{s-1}}{}_{q_s}+2\overset{s+1}{H}_{u_0|r_s|}{}^{r_{s+1}}\overset{s+1}{H}_{\sigma_0]q_sr_{s+1}}\Big)\\
&+\overset{s+1}{H}_{[r_0|q_s|}{}^{r_{s+1}}\Big(\overset{0}{B}{}^{\rho\mu}_{u_0\sigma_0]}\overset{s+1}{B}{}^{\lambda}_{r_{s+1}}\overset{s-1}{B}{}^{\kappa}_{p_{s-1}}K_{\rho\mu\lambda\kappa}-2H_{u_0|p_{s-1}r_s|}\overset{s+1}{H}_{\sigma_0]}{}^{r_s}{}_{r_{s+1}}\Big)\\
&-\overset{s-1}{H}_{[r_0\cdots r_{s-2}|p_{s-1}|}\Big(\overset{0}{B}{}^{\rho\mu}_{u_0\sigma_0]}\overset{s}{B}{}^{\lambda}_{q_s}\overset{s-2}{B}{}^{\kappa}_{r_{s-2}}K_{\rho\mu\lambda\kappa}-2H_{u_0|r_{s-2}r_{s-1}|}\overset{s-1}{H}_{\sigma_0]}{}^{r_{s-1}}{}_{q_s}\Big)\\
&=0.
\end{aligned}
\tag{3.12}
$$

[25]

From (3.12) we conclude that if the Gauss equation $\overset{s-1}{N}$ and the Ricci equations $\overset{s+1}{N}_{s-1}$, $\overset{s}{N}_{s-2}$ hold, then

$$
\begin{aligned}
&\overset{s}{H}_{[r_0|p_{s-1}|}{}^{r_s}\Big(\overset{0}{B}{}^{\rho\mu}_{u_0\sigma_0]}\overset{s}{B}{}^{\lambda}_{q_s}\overset{s}{B}{}^{\kappa}_{r_s}K_{\rho\mu\lambda\kappa}-\overset{s}{K}_{u_0\sigma_0]q_sr_s}\\
&+2H_{u_0|r_{s-1}r_s|}\overset{s}{H}_{\sigma_0]}{}^{r_{s-1}}{}_{q_s}+2\overset{s+1}{H}_{u_0|r_s|}{}^{r_{s+1}}\overset{s+1}{H}_{\sigma_0]q_sr_{s+1}}\Big)=0.
\end{aligned}
\tag{3.13}
$$

If we can show from (3.13) that $\theta_{u_0\sigma_0q_sr_s}$ vanishes, where

$$
\begin{aligned}
\theta_{u_0\sigma_0q_sr_s}&=\overset{0}{B}{}^{\rho\mu}_{u_0\sigma_0}\overset{s}{B}{}^{\lambda}_{q_s}\overset{s}{B}{}^{\kappa}_{r_s}K_{\rho\mu\lambda\kappa}-\overset{s}{K}_{u_0\sigma_0q_sr_s}\\
&+2H_{[u_0|r_{s-1}r_s|}\overset{s}{H}_{\sigma_0]}{}^{r_{s-1}}{}_{q_s}+2\overset{s+1}{H}_{[u_0|r_s|}{}^{r_{s+1}}\overset{s+1}{H}_{\sigma_0]q_sr_{s+1}},
\end{aligned}
\tag{3.14}
$$

then we will have shown that $\overset{s}{N}$ follows from $\overset{s}{N}_{s-1}$, $\overset{s-1}{N}_{s-1}$, $\overset{s}{N}_{s-2}$, $\overset{s+1}{N}_{s-1}$.

Using the definition of $\theta_{u_0\sigma_0q_sr_s}$ given by (3.14), we can rewrite (3.13) as

$$
\overset{s}{H}_{[r_0|p_{s-1}|}{}^{r_s}\theta_{u_0\sigma_0]q_sr_s}=0. \tag{3.15}
$$

But by its very definition, θ is alternating in its first two indices. For, by (3.14), (1.17), and the First Identity for the Riemann-Christoffel affinor,

$$
K_{[\rho\mu]\lambda\kappa}=K_{\rho\mu\lambda\kappa}, \tag{3.16}
$$

[26]

[25] In arriving at the first term of (3.12) we used the Third Identity for the Riemann-Christoffel affinor, $K_{\rho\mu\lambda\kappa}=-K_{\rho\mu\kappa\lambda}$. See I, pg. 113.

[26] I, pg. 113.

we have

$$\begin{aligned}\theta_{[u_0s_0]q_sr_s} &= \overset{0}{B}{}^{r_s}_{[u_0s_0]}\dot{B}^{\lambda}_{q_s}\overset{s}{B}{}^{\kappa}_{r_s}K_{r\mu\lambda\kappa} - \overset{s}{K}_{[u_0s_0]q_sr_s} \\ &\quad + 2\overset{s}{H}_{[u_0|r_{s-1}r_s|}\dot{H}_{s_0]}{}^{r_{s-1}}{}_{q_s} + 2\overset{s+1}{H}_{[u_0|r_s|}\dot{H}^{r_{s+1}}_{s_0]q_sr_{s+1}} \\ &= \overset{0}{B}{}^{r_s}_{u_0s_0}\overset{s}{B}{}^{\lambda'}_{q_s}\overset{s}{B}{}^{\kappa}_{r_s}K_{r\mu\lambda\kappa} - \overset{s}{K}_{u_0s_0q_sr_s} + 2\overset{s}{H}_{[u_0|r_{s-1}r_s|}\dot{H}_{s_0]}{}^{r_{s-1}}{}_{q_s} \\ &\quad + 2\overset{s+1}{H}_{[u_0|r_s|}\dot{H}^{r_{s+1}}_{s_0]q_sr_{s+1}} = \theta_{u_0s_0q_sr_s}.\end{aligned} \tag{3.17}$$

Therefore, we can write (3.15) as

$$\overset{s}{H}_{r_0p_{s-1}}\dot{}^{r_s}\theta_{u_0s_0q_sr_s} + \overset{s}{H}_{u_0p_{s-1}}\dot{}^{r_s}\theta_{s_0r_0q_sr_s} + \overset{s}{H}_{s_0p_{s-1}}\dot{}^{r_s}\theta_{r_0u_0q_sr_s} = 0. \tag{3.18}$$

Now our goal is to show from (3.18) that $\theta_{u_0s_0q_sr_s}$ vanishes.

To do this we assume that condition 0 for $\overset{s}{H}$ is satisfied, and then we make use of the vectors $\overset{s}{u}{}^{r_0}_{A}$, $\overset{s}{v}{}^{r_0}_{B}$, $\overset{s}{z}{}^{p_{s-1}}_{B}$ that we defined in Section II. Let us transvect in (3.18) by $\overset{s}{u}{}^{s_0}_{A_1}\overset{s}{u}{}^{r_0}_{A_2}$. We then have

$$\overset{s}{H}_{u_0p_{s-1}}\dot{}^{r_s}\overset{s}{u}{}^{s_0}_{A_1}\overset{s}{u}{}^{r_0}_{A_2}\theta_{s_0r_0q_sr_s} = 0 \tag{3.19}$$

where A_1 and A_2 can be chosen arbitrarily. We can consider A_1, A_2, q_s fixed once arbitrarily chosen, and then under the assumption that condition 0 for $\overset{s}{H}$ is satisfied, (3.19) tells us that the vector $\overset{s}{u}{}^{s_0}_{A_1}\overset{s}{u}{}^{r_0}_{A_2}\theta_{s_0r_0q_sr_s}$ of the x^{th} normal space is normal to the set of vectors $\overset{s}{v}{}^{u_0}_{B}\overset{s}{z}{}^{p_{s-1}}_{B}\overset{s}{H}_{u_0p_{s-1}}\dot{}^{r_s}$, all of which also lie in the x^{th} normal space. Since there are m independent vectors in that set, we can conclude that

$$\overset{s}{u}{}^{s_0}_{A_1}\overset{s}{u}{}^{r_0}_{A_2}\theta_{s_0r_0q_sr_s} = 0, \tag{3.20}$$

where A_1 and A_2 can be chosen arbitrarily.

If we transvect in (3.18) by $\overset{s}{u}{}^{s_0}_{A_1}\overset{s}{v}{}^{r_0}_{B_1}\overset{s}{z}{}^{p_{s-1}}_{B_2}$, we arrive at

$$\overset{s}{z}{}^{p_{s-1}}_{B_2}\overset{s}{H}_{u_0p_{s-1}}\dot{}^{r_s}\overset{s}{u}{}^{s_0}_{A_1}\overset{s}{v}{}^{r_0}_{B_1}\theta_{s_0r_0q_sr_s} = 0, \tag{3.21}$$

where A_1 and B_1 can be chosen arbitrarily, but where $B_2 \neq B_1$. If we assume that condition 1 for $\overset{z}{H}$ is satisfied, then we can conclude from (3.21) and from (3.17) that

$$\overset{z}{u}{}^{s_0}_{A_1}\overset{z}{v}{}^{r_0}_{B_1}\theta_{s_0r_0q_zr_z} = -\overset{z}{v}{}^{s_0}_{B_1}\overset{z}{u}{}^{r_0}_{A_1}\theta_{s_0r_0q_zr_z} = 0, \tag{3.22}$$

where A_1 and B_1 can be chosen arbitrarily. For, under that assumption, considering q_z, A_1, and B_1 fixed once arbitrarily chosen, (3.21) tells us that the vector $\overset{z}{u}{}^{s_0}_{A_1}\overset{z}{v}{}^{r_0}_{B_1}\theta_{s_0r_0q_zr_z}$ of the x^{th} normal space is normal to the m independent vectors which lie in the x^{th} normal space and are included in the set of $\beta - 1$ vectors $\overset{z}{v}{}^{u_0}_{B}\overset{z}{z}{}^{p_z-1}_{B}\overset{z}{H}_{u_0p_z-1}\dot{}\,\dot{}^{r_z}$, $B \neq B_1$.

Finally, if we transvect in (3.18) by $\overset{z}{v}{}^{u_0}_{B_1}\overset{z}{v}{}^{s_0}_{B_2}\overset{z}{v}{}^{r_0}_{B_3}\overset{z}{z}{}^{p_z-1}_{B_1}$, we have

$$\overset{z}{v}{}^{u_0}_{B_1}\overset{z}{z}{}^{p_z-1}_{B_1}\overset{z}{H}_{u_0p_z-1}\dot{}\,\dot{}^{r_z}\overset{z}{v}{}^{s_0}_{B_2}\overset{z}{v}{}^{r_0}_{B_3}\theta_{s_0r_0q_zr_z} = 0 \tag{3.23}$$

where B_2 and B_3 can be chosen arbitrarily, but where $B_1 \neq B_2, B_3$. If we assume that condition 2 for $\overset{z}{H}$ is satisfied, and if B_2, B_3, q_z be considered fixed once arbitrarily chosen, (3.23) tells us that the vector $\overset{z}{v}{}^{s_0}_{B_2}\overset{z}{v}{}^{r_0}_{B_3}\theta_{s_0r_0q_zr_z}$ of the x^{th} normal space is normal to the m independent vectors which are included in the set $\overset{z}{v}{}^{u_0}_{B}\overset{z}{z}{}^{p_z-1}_{B}\overset{z}{H}_{u_0p_z-1}\dot{}\,\dot{}^{r_z}$, $B \neq B_2, B_3$, and which also lie in the x^{th} normal space. Therefore, under that condition,

$$\overset{z}{v}{}^{s_0}_{B_2}\overset{z}{v}{}^{r_0}_{B_3}\theta_{s_0r_0q_zr_z} = 0, \tag{3.24}$$

where B_2 and B_3 can be chosen arbitrarily.

Now, when we recall that $\overset{z}{v}{}^{r_0}_{B}$ and $\overset{z}{u}{}^{r_0}_{A}$ constitute a set of m independent vectors of the V_m, and when we consider (3.20), (3.22), and (3.24) we can conclude that

THEOREM IIIA. If condition 2 for $\overset{z}{H}$ is satisfied, $x = 1, \cdots, k$, then the Gauss equation $\overset{z}{N}$ is a consequence of the Codazzi equation

$\overset{s}{N}_{s-1}$, the Gauss equation $\overset{s-1}{N}_{s-1}$, and the Ricci equations $\overset{s}{N}_{s-2}$, $\overset{s+1}{N}_{s-1}$; or, symbolically stated, $\overset{s}{N}_{s-1}, \overset{s-1}{N}_{s-1}, \overset{s}{N}_{s-2}, \overset{s+1}{N}_{s-1} \to \overset{s}{N}$.

B. $\overset{s}{N}_{s-1}, \overset{s-1}{N}_{s-1}, \overset{s}{N}_{s}, \overset{s}{N}_{s-2}, \to \overset{s+1}{N}_{s-1}$.

The formal work leading to (3.12) suggests to us another theorem on the dependence of the Gauss-Codazzi-Ricci equations on each other. If instead of making the assumptions that led to (3.18) we were to assume that the Gauss equations $\overset{s-1}{N}_{s-1}$, $\overset{s}{N}_{s}$ and the Ricci equation $\overset{s}{N}_{s-2}$ held, then from (3.12) we would conclude that

$$(3.25)\quad \overset{s+1}{H}_{[r_0|q_s|}{}^{r_{s+1}}(\overset{0}{B}{}^{\rho\mu}_{u_0s_0]}\overset{s+1}{B}{}^{\lambda}_{r_{s+1}}\overset{s-1}{B}{}^{\kappa}_{p_{s-1}}K_{\rho\mu\lambda\kappa} - 2\overset{s}{H}_{u_0|p_{s-1}r_s|}\overset{s+1}{H}{}^{r_s}_{s_0]}{}^{r_{s+1}}) = 0.$$

If we can show from (3.25) that $\theta_{u_0s_0r_{s+1}p_{s-1}}$ vanishes, where

$$(3.26)\quad \theta_{u_0s_0r_{s+1}p_{s-1}} = \overset{0}{B}{}^{\rho\mu}_{u_0s_0}\overset{s+1}{B}{}^{\lambda}_{r_{s+1}}\overset{s-1}{B}{}^{\kappa}_{p_{s-1}}K_{\rho\mu\lambda\kappa} - 2\overset{s}{H}_{[u_0|p_{s-1}r_s|}\overset{s+1}{H}{}^{r_s}_{s_0]}{}^{r_{s+1}}$$

then we will have shown that $\overset{s+1}{N}_{s-1}$ follows from $\overset{s}{N}_{s-1}, \overset{s-1}{N}_{s-1}, \overset{s}{N}_{s}, \overset{s}{N}_{s-2}$. From (3.26) and the First Identity for the Riemann-Christoffel affinor, (3.16), it is clear that

$$(3.27)\quad \theta_{[u_0s_0]r_{s+1}p_{s-1}} = \theta_{u_0s_0r_{s+1}p_{s-1}},$$

and therefore we can write (3.25) as

$$(3.28)\quad \begin{aligned}\overset{s+1}{H}_{r_0q_s}{}^{r_{s+1}}\theta_{u_0s_0r_{s+1}p_{s-1}} + \overset{s+1}{H}_{u_0q_s}{}^{r_{s+1}}\theta_{s_0r_0r_{s+1}p_{s-1}} \\ + \overset{s+1}{H}_{s_0q_s}{}^{r_{s+1}}\theta_{r_0u_0r_{s+1}p_{s-1}} = 0.\end{aligned}$$

Our goal is to show that $\theta_{u_0s_0r_{s+1}p_{s-1}}$ vanishes.

As in the proof of Theorem III A, we assume that condition 0 for $\overset{s+1}{H}$ is satisfied, and we use the vectors $\overset{s+1}{u}{}^{r_0}_{A}$, $\overset{s+1}{v}{}^{r_0}_{B}$, $\overset{s+1}{z}{}^{p_s}_{B}$ thus defined. If we transvect in (3.28) by $\overset{s+1}{u}{}^{s_0}_{A_1}$, $\overset{s+1}{u}{}^{r_0}_{A_2}$, we have

$$(3.29)\quad \overset{s+1}{H}_{u_0q_s}{}^{r_{s+1}}\overset{s+1}{u}{}^{s_0}_{A_1}\overset{s+1}{u}{}^{r_0}_{A_2}\theta_{s_0r_0r_{s+1}p_{s-1}} = 0$$

where A_1 and A_2 can be chosen arbitrarily. Because condition 0 for $\overset{x+1}{H}$ is satisfied, (3.29) tells us that the vector $\underset{A_1\ A_2}{\overset{x+1\ x+1}{u^{\alpha_0}\ u^{\gamma_0}}}\theta_{\alpha_0\gamma_0\gamma_{x+1}p_{x-1}}$, which lies in the $(x+1)^{st}$ normal space (A_1, A_2, p_{x-1} being considered fixed once arbitrarily chosen), is normal to the m independent vectors which are included in the set of $\underset{x+1}{\beta}$ vectors $\underset{B}{\overset{x+1}{v^{\omega_0}}}\ \underset{B}{\overset{x+1\ x+1}{z^{\alpha_x}}}H^{\cdot\cdot\cdot\gamma_{x+1}}_{\omega_0\alpha_x}$ and which also lie in the $(x+1)^{st}$ normal space. Hence, we conclude that if condition 0 for $\overset{x+1}{H}{}^{\cdot\cdot\cdot\gamma_{x+1}}_{\omega_0\alpha_x}$ is satisfied,

$$\underset{A_1\ A_2}{\overset{x+1\ x+1}{u^{\alpha_0}u^{\gamma_0}}}\theta_{\alpha_0\gamma_0\gamma_{x+1}p_{x-1}} = 0, \tag{3.30}$$

where A_1 and A_2 can be chosen arbitrarily.

We continue as before, and we transvect in (3.28) by $\underset{A_1\ B_1\ B_2}{\overset{x+1\ x+1\ x+1}{u^{\alpha_0}\ v^{\gamma_0}\ z^{\alpha_x}}}$.

$$\underset{B_2}{\overset{x+1}{z^{\alpha_x}}}H^{\cdot\cdot\cdot\gamma_{x+1}}_{\omega_0\alpha_x}\ \underset{A_1\ B_1}{\overset{x+1\ x+1}{u^{\alpha_0}\ v^{\gamma_0}}}\theta_{\alpha_0\gamma_0\gamma_{x+1}p_{x-1}} = 0, \tag{3.31}$$

where A_1 and B_1 can be chosen arbitrarily, but where $B_2 \neq B_1$. We see that if condition 1 for $\overset{x+1}{H}$ is satisfied, then (3.31) tells us that the vector $\underset{A_1\ B_1}{\overset{x+1\ x+1}{u^{\alpha_0}\ v^{\gamma_{x+1}}}}\theta_{\alpha_0\gamma_0\gamma_{x+1}p_{x-1}}$, which lies in the $(x+1)^{st}$ normal space (A_1, B_1, p_{x-1} considered fixed once arbitrarily chosen), is normal to the $\underset{x+1}{m}$ independent vectors which are included in the set of $\underset{x+1}{\beta} - 1$ vectors $\underset{B}{\overset{x+1}{v^{\omega_0}}}\ \underset{B}{\overset{x+1\ x+1}{z^{\alpha_x}}}H^{\cdot\cdot\cdot\gamma_{x+1}}_{\omega_0\alpha_x}$, $B \neq B_1$, and which also lie in the $(x+1)^{st}$ normal space. Therefore we can say that if condition 1 for $\overset{x+1}{H}$ is satisfied, (3.31) and (3.27) tell us that

$$\underset{A_1\ B_1}{\overset{x+1\ x+1}{u^{\alpha_0}\ v^{\gamma_0}}}\theta_{\alpha_0\gamma_0\gamma_{x+1}p_{x-1}} = -\underset{B_1\ A_1}{\overset{x+1\ x+1}{v^{\alpha_0}\ u^{\gamma_0}}}\theta_{\alpha_0\gamma_0\gamma_{x+1}p_{x-1}} = 0, \tag{3.32}$$

where A_1 and B_1 can be chosen arbitrarily.

Just as in the work done for Theorem III A, we now transvect in (3.28) by $\underset{B_1\ B_2\ B_3\ B_1}{\overset{x+1\ x+1\ x+1\ x+1}{v^{\omega_0}\ v^{\alpha_0}\ v^{\gamma_0}\ z^{\alpha_x}}}$, and we have

$$\underset{B_1\ B_1}{\overset{x+1\ x+1}{v^{\omega_0}\ z^{\alpha_x}}}H^{\cdot\cdot\cdot\gamma_{x+1}}_{\omega_0\alpha_x}\ \underset{B_2\ B_3}{\overset{x+1\ x+1}{v^{\alpha_0}\ v^{\gamma_0}}}\theta_{\alpha_0\gamma_0\gamma_{x+1}p_{x-1}} = 0, \tag{3.33}$$

where B_2 and B_3 can be chosen arbitrarily, but where $B_1 \neq B_2, B_3$. (3.33) tells us that if condition 2 for $\overset{x+1}{H}$ is satisfied, the vector $\underset{B_2}{\overset{x+1}{v^{s_0}}}\ \underset{B_3}{\overset{x+1}{v^{r_0}}}\theta_{s_0r_0r_{x+1}p_{x-1}}$ (B_2, B_3, p_{x-1} considered fixed once arbitrarily chosen), which lies in the $(x+1)^{st}$ normal space, is normal to the $\underset{x+1}{m}$ independent vectors which are included in the set of $\underset{x+1}{\beta} - 2$ vectors $\underset{B}{\overset{x+1}{v^{u_0}}}\ \underset{B}{\overset{x+1}{z^{q_x}}}H_{u_0q_x}{}^{r_{x+1}}$, $B \neq B_2, B_3$, and which also lie in the $(x+1)^{st}$ normal space. Therefore, if condition 2 for $\overset{x+1}{H}$ is satisfied, then

$$\underset{B_2}{\overset{x+1}{v^{s_0}}}\ \underset{B_3}{\overset{x+1}{v^{r_0}}}\theta_{s_0r_0r_{x+1}p_{x-1}} = 0, \tag{3.34}$$

where B_2 and B_3 can be chosen arbitrarily.

But now, considering (3.30), (3.32), (3.34) and the fact that $\underset{A}{\overset{x+1}{u^{r_0}}}$ and $\underset{B}{\overset{x+1}{v^{r_0}}}$ constitute a set of $\underset{0}{m}$ independent vectors of the V_m, we conclude that

THEOREM III B. If condition 2 for $\overset{x+1}{H}$ is satisfied, $x = 1, \cdots, k$, then the Ricci equation $\underset{x-1}{\overset{x+1}{N}}$ follows from the Codazzi equation $\underset{x-1}{\overset{x}{N}}$, the Gauss equations $\underset{x-1}{\overset{x-1}{N}}, \underset{x}{\overset{x}{N}}$, and the Ricci equation $\underset{x-2}{\overset{x}{N}}$; or, symbolically stated, $\underset{x-1}{\overset{x}{N}}, \underset{x-1}{\overset{x-1}{N}}, \underset{x}{\overset{x}{N}}, \underset{x-2}{\overset{x}{N}} \rightarrow \underset{x-1}{\overset{x+1}{N}}$.

IV. Relationships Arrived at by Working with the Equations $\underset{x-g}{\overset{x}{N}}, g = 2, \cdots, x$

We could repeat the work of Section III step by step now, starting with any equation $\underset{x-g}{\overset{x}{N}}, g = 2, \cdots, x; x = g, \cdots, k$, instead of with the Codazzi equation $\underset{x-1}{\overset{x}{N}}$. We could differentiate, simplify by using the Frenet equations, and alternate on three indices just as we did before. Then by assuming that the equations $\underset{x-g}{\overset{x-1}{N}}, \underset{x-g-1}{\overset{x}{N}}, \underset{x-g}{\overset{x+1}{N}}$ also held, which

assumptions are exactly analagous to those we made in the discussion preceding (3.13), we would arrive at an equation exactly analagous to (3.13), and hence at another equation exactly analagous to (3.18). From this latter equation we could conclude just as we did in Section III that the equation $\underset{x-g+1}{\overset{x}{N}}$ holds if condition 2 for $\underset{x-g+1}{\overset{x-g+1}{H}}$ is satisfied. The statement of our theorem would then be

THEOREM A. If condition 2 for $\underset{x-g+1}{\overset{x-g+1}{H}}$ is satisfied, $g = 1, \cdots, x$, $x = g, \cdots, k$, then the equation $\underset{x-g+1}{\overset{x}{N}}$ is a consequence of the equations $\underset{x-g}{\overset{x}{N}}, \underset{x-g}{\overset{x-1}{N}}, \underset{x-g-1}{\overset{x}{N}}, \underset{x-g}{\overset{x+1}{N}}$. Or, symbolically stated, $\underset{x-g}{\overset{x}{N}}, \underset{x-g}{\overset{x-1}{N}}, \underset{x-g-1}{\overset{x}{N}}, \underset{x-g}{\overset{x+1}{N}} \rightarrow \underset{x-g+1}{\overset{x}{N}}$.[27]

A theorem exactly analagous to Theorem III B is arrived at if in the process outlined above one assumes that the equation $\underset{x-g+1}{\overset{x}{N}}$ holds rather than the equation $\underset{x-g}{\overset{x+1}{N}}$. For then we obtain an equation analagous to (3.25), and from that equation one exactly analagous to (3.28). By reasoning word for word the same as that following (3.28) we can then conclude that the equation $\underset{x-g}{\overset{x+1}{N}}$ holds if condition 2 for $\overset{x+1}{H}$ is satisfied. The statement of this theorem would be

THEOREM B. If condition 2 for $\overset{x+1}{H}$ is satisfied, $g = 1, \cdots, x$, $x = g, \cdots, k$, then the equation $\underset{x-g}{\overset{x+1}{N}}$ is a consequence of the equations $\underset{x-g}{\overset{x}{N}}, \underset{x-g}{\overset{x-1}{N}}, \underset{x-g+1}{\overset{x}{N}}, \underset{x-g-1}{\overset{x}{N}}$. Or, symbolically stated, $\underset{x-g}{\overset{x}{N}}, \underset{x-g}{\overset{x-1}{N}}, \underset{x-g+1}{\overset{x}{N}}, \underset{x-g-1}{\overset{x}{N}} \rightarrow \underset{x-g}{\overset{x+1}{N}}$.[28,29]

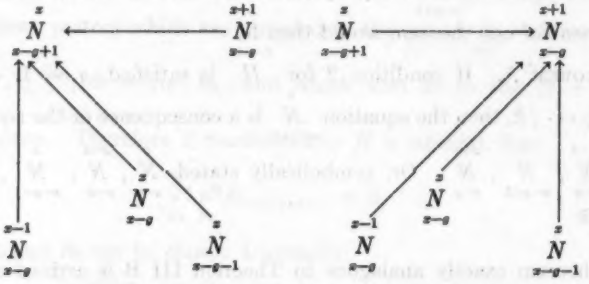
Theorems A and B may perhaps be expressed to advantage in still

[27] We have written $g = 1, \cdots, x$ rather than $g = 2, \cdots, x$ because when $g = 1$, Theorem A is exactly Theorem III A. The theorem therefore holds in that case also.

[28] We have written $g = 1, \cdots, x$ rather than $g = 2, \cdots, x$ because when $g = 1$ Theorem B is exactly Theorem III B. The theorem therefore holds in that case also.

[29] For a direct demonstration of Theorems A and B see A. Schwartz, Mathematics Thesis, Massachusetts Institute of Technology, 1939.

another schematic way. If $Q \overset{R}{\underset{S}{\rightarrow}} P$ be taken to mean that P is a consequence of Q, R, and S, the following diagrams give the statements of Theorems A and B:



Theorem A Theorem B

It is interesting to note the "symmetry" of these diagrams.

V. The Relationship Arrived at by Working with the Gauss Equation $\overset{x}{N}_{x}$

We cannot expect to derive theorems analagous to Theorems A and B by working here with $\overset{x}{N}_{x}$ as we did above with $\overset{x}{N}_{x-g}$, $g = 1, \cdots, x$, because the result we would hope to arrive at would be meaningless.

Notice that equations $\overset{x}{N}_{x-1}$ and $\overset{x-1}{N}_{x}$ are the same equation. This is true because $\overset{x-1}{N}_{x}$ can be written as

$$B^{r\mu}_{s_0 r_0} \overset{0}{B}{}^{\lambda}_{q_{x-1}} \overset{x-1}{B}{}^{x}_{p_x} K_{r\mu\lambda x} = +\, 2D_{[s_0} \overset{x}{H}_{r_0]q_{x-1}p_x}, \tag{5.1}$$

[30]

and because the Third Identity for the Riemann-Christoffel affinor,

$$K_{r\mu\lambda x} = -\, K_{r\mu x\lambda}, \tag{5.2}$$

[31]

enables us to put $\overset{x-1}{N}_{x}$ into the form

$$-\, B^{r\mu}_{s_0 r_0} \overset{0}{B}{}^{x}_{p_x} \overset{x}{B}{}^{\lambda}_{q_{x-1}} K_{r\mu\lambda x} = +\, 2D_{[s_0} \overset{x}{H}_{r_0]q_{x-1}p_x}. \tag{5.3}$$

[30] II, pg. 123, (13.36).

[31] I, pg. 113.

This form of the equation $\overset{x-1}{N}$ is clearly the Codazzi equation $\overset{x}{N}$. But when $\overset{x+1}{N} \equiv \overset{x}{N}$, the statement $\overset{x}{N}, \overset{x-1}{N}, \overset{x}{N}, \overset{x+1}{N} \to \overset{x}{N}$, which is what we would expect if we could parallel Theorem A, becomes meaningless.

However, even though we do not expect theorems here which will fit into our general scheme, we do have a theorem which demonstrates a relationship between the Gauss equation $\overset{x}{N}$ and the Codazzi equations $\overset{x}{N}$, $\overset{x+1}{N}$. The conditions under which this result holds are quite naturally stronger conditions than those we have in Theorems A and B.

A. $\overset{x}{N}, \overset{x}{N} \to \overset{x+1}{N}$.

The Gauss equation $\overset{x}{N}$, when written out, is

$$\begin{aligned}(5.4)\qquad & \overset{0}{B}{}^{\rho\mu}_{s_0r_0}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x}{B}{}^{\kappa}_{p_x}K_{\rho\mu\lambda\kappa} = \overset{x}{K}_{s_0r_0q_xp_x} - 2\overset{x}{H}_{[s_0|r_{x-1}p_x|}\overset{x}{H}_{r_0]}{}^{r_{x-1}}{}_{q_x} \\ & - 2\overset{x+1}{H}_{[s_0|p_xr_{x+1}|}\overset{x+1}{H}_{r_0]q_x}{}^{r_{x+1}}, \qquad x = 0, 1, \cdots, k.\end{aligned}$$

As before, we differentiate, and then change the form of our left hand member by using the Frenet equations (1.9) and (1.10).

$$\begin{aligned}(5.5)\qquad & (D_{u_0}\overset{0}{B}{}^{\rho}_{s_0})\overset{0}{B}{}^{\rho}_{r_0}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x}{B}{}^{\kappa}_{p_x}K_{\rho\mu\lambda\kappa} + (D_{u_0}\overset{0}{B}{}^{\rho}_{r_0})\overset{0}{B}{}^{\rho}_{s_0}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x}{B}{}^{\kappa}_{p_x}K_{\rho\mu\lambda\kappa} \\ & + (D_{u_0}\overset{x}{B}{}^{\lambda}_{q_x})\overset{0}{B}{}^{\rho\mu}_{s_0r_0}\overset{x}{B}{}^{\kappa}_{p_x}K_{\rho\mu\lambda\kappa} + (D_{u_0}\overset{x}{B}{}^{\kappa}_{p_x})\overset{0}{B}{}^{\rho\mu}_{s_0r_0}\overset{x}{B}{}^{\lambda}_{q_x}K_{\rho\mu\lambda\kappa} \\ & + \overset{0}{B}{}^{\rho\mu}_{s_0r_0}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x}{B}{}^{\kappa}_{p_x}D_{u_0}K_{\rho\mu\lambda\kappa} = D_{u_0}\overset{x}{K}_{s_0r_0q_xp_x} \\ & - 2\overset{x}{H}_{[s_0|r_{x-1}p_x|}D_{|u_0|}\overset{x}{H}_{r_0]}{}^{r_{x-1}}{}_{q_x} - 2\overset{x}{H}_{[r_0}{}^{r_{x-1}}{}_{|q_x|}D_{|u_0|}\overset{x}{H}_{s_0]r_{x-1}p_x} \\ & - 2\overset{x+1}{H}_{[s_0|p_xr_{x+1}|}D_{|u_0|}\overset{x+1}{H}_{r_0]q_x}{}^{r_{x+1}} - 2\overset{x+1}{H}_{[r_0|q_x|}{}^{r_{x+1}}D_{|u_0|}\overset{x+1}{H}_{s_0]p_xr_{x+1}}\cdot\end{aligned}$$

$$\begin{aligned}(5.6)\qquad & \overset{1}{H}_{u_0s_0}{}^{\rho}\overset{0}{B}{}^{\rho}_{r_0}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x}{B}{}^{\kappa}_{p_x}K_{\rho\mu\lambda\kappa} + \overset{1}{H}_{u_0r_0}{}^{\mu}\overset{0}{B}{}^{\rho}_{s_0}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x}{B}{}^{\kappa}_{p_x}K_{\rho\mu\lambda\kappa} \\ & + (-\overset{x}{H}_{u_0}{}^{\lambda}{}_{q_x} + \overset{x+1}{H}_{u_0q_x}{}^{\lambda})\overset{0}{B}{}^{\rho\mu}_{s_0r_0}\overset{x}{B}{}^{\kappa}_{p_x}K_{\rho\mu\lambda\kappa} \\ & + (-\overset{x}{H}_{u_0}{}^{\kappa}{}_{p_x} + \overset{x+1}{H}_{u_0p_x}{}^{\kappa})\overset{0}{B}{}^{\rho\mu}_{s_0r_0}\overset{x}{B}{}^{\lambda}_{q_x}K_{\rho\mu\lambda\kappa} + \overset{0}{B}{}^{\rho\mu}_{s_0r_0}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x}{B}{}^{\kappa}_{p_x}D_{u_0}K_{\rho\mu\lambda\kappa} \\ & = D_{u_0}\overset{x}{K}_{s_0r_0q_xp_x} - 2\overset{x}{H}_{[s_0|r_{x-1}p_x|}D_{|u_0|}\overset{x}{H}_{r_0]}{}^{r_{x-1}}{}_{q_x} \\ & - 2\overset{x}{H}_{[r_0}{}^{r_{x-1}}{}_{|q_x|}D_{|u_0|}\overset{x}{H}_{s_0]r_{x-1}p_x} - 2\overset{x+1}{H}_{[s_0|p_xr_{x+1}|}D_{|u_0|}\overset{x+1}{H}_{r_0]q_x}{}^{r_{x+1}} \\ & - 2\overset{x+1}{H}_{[r_0|q_x|}{}^{r_{x+1}}D_{|u_0|}\overset{x+1}{H}_{s_0]p_xr_{x+1}}\cdot\end{aligned}$$

After alternation on the indices u_0, s_0, r_0, (5.6) becomes

$$
\begin{aligned}
&\overset{1}{H}_{[u_0 s_0}{}^{\cdot\cdot\cdot\nu}\overset{0}{B}{}^{\rho\mu}_{r_0]}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x}{B}{}^{\kappa}_{p_x}K_{\nu\mu\lambda\kappa}+\overset{1}{H}_{[u_0 r_0}{}^{\cdot\cdot\cdot\mu}\overset{0}{B}{}^{\nu\rho}_{s_0]}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x}{B}{}^{\kappa}_{p_x}K_{\nu\mu\lambda\kappa}\\
&\quad+(-\overset{x}{H}_{[u_0}{}^{\cdot\cdot\lambda}{}_{|q_x|}+\overset{x+1}{H}_{[u_0|}{}^{\cdot\cdot\cdot\lambda}{}_{q_x|})\overset{0}{B}{}^{\nu\rho}_{s_0 r_0]}\overset{x}{B}{}^{\kappa}_{p_x}K_{\nu\mu\lambda\kappa}\\
&\quad+(-\overset{x}{H}_{[u_0}{}^{\cdot\cdot\kappa}{}_{|p_x|}+\overset{x+1}{H}_{[u_0|}{}^{\cdot\cdot\cdot\kappa}{}_{p_x|})\overset{0}{B}{}^{\nu\rho}_{s_0 r_0]}\overset{x}{B}{}^{\lambda}_{q_x}K_{\nu\mu\lambda\kappa}\\
&\quad+\overset{0}{B}{}^{\nu\rho}_{[s_0 r_0}\overset{x}{B}{}^{\lambda}_{|q_x|}\overset{x}{B}{}^{\kappa}_{|p_x|}D_{u_0]}K_{\nu\mu\lambda\kappa}=\overset{x}{D}_{[u_0}K_{s_0 r_0]q_x p_x}\\
&\quad-2\overset{x}{H}_{[s_0|r_{x-1}p_x|}D_{u_0}\overset{x}{H}_{r_0]}{}^{\cdot\cdot\cdot r_{x-1}}{}_{q_x}-2\overset{x}{H}_{[r_0}{}^{\cdot\cdot\cdot r_{x-1}}{}_{|q_x|}D_{u_0}\overset{x}{H}_{s_0]r_{x-1}p_x}\\
&\quad-2\overset{x+1}{H}_{[s_0|p_x r_{x+1}|}D_{u_0}\overset{x+1}{H}_{r_0]}{}^{\cdot\cdot\cdot r_{x+1}}{}_{q_x}-2\overset{x+1}{H}_{[r_0|q_x|}{}^{\cdot\cdot\cdot r_{x+1}}D_{u_0}\overset{x+1}{H}_{s_0]p_x r_{x+1}}.
\end{aligned}
\tag{5.7}
$$

But (5.7) can be simplified because many of its terms vanish. For instance, the first two terms vanish because $\overset{1}{H}_{[u_0 s_0]}{}^{\cdot\cdot\cdot\nu}=0$;[33] while the last term of the left hand member vanishes because of the Bianchi Identity for the Riemann-Christoffel affinor.[33] The first term of the right hand member vanishes because of the identity for $\overset{x}{K}_{s_0 r_0 q_x p_x}$ which is analagous to the Bianchi Identity.[34] If now the relationships (3.6) and (3.7) be borne in mind, we can write (5.7) as

$$
\begin{aligned}
&-\overset{x}{H}_{[u_0}{}^{\cdot\cdot\cdot r_{x-1}}{}_{|q_x|}(\overset{0}{B}{}^{\nu\rho}_{s_0 r_0]}\overset{x-1}{B}{}^{\lambda}_{r_{x-1}}\overset{x}{B}{}^{\kappa}_{p_x}K_{\nu\mu\lambda\kappa}-2D_{s_0}\overset{x}{H}_{r_0]r_{x-1}p_x})\\
&\quad+\overset{x+1}{H}_{[u_0|q_x|}{}^{\cdot\cdot\cdot r_{x+1}}(\overset{0}{B}{}^{\nu\rho}_{s_0 r_0]}\overset{x+1}{B}{}^{\lambda}_{r_{x+1}}\overset{x}{B}{}^{\kappa}_{p_x}K_{\nu\mu\lambda\kappa}+2D_{s_0}\overset{x+1}{H}_{r_0]p_x r_{x+1}})\\
&\quad-\overset{x}{H}_{[u_0}{}^{\cdot\cdot\cdot r_{x-1}}{}_{|p_x|}(\overset{0}{B}{}^{\nu\rho}_{s_0 r_0]}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x-1}{B}{}^{\kappa}_{r_{x-1}}K_{\nu\mu\lambda\kappa}+2D_{s_0}\overset{x}{H}_{r_0]r_{x-1}q_x})\\
&\quad+\overset{x+1}{H}_{[u_0|p_x|}{}^{\cdot\cdot\cdot r_{x+1}}(\overset{0}{B}{}^{\nu\rho}_{s_0 r_0]}\overset{x}{B}{}^{\lambda}_{q_x}\overset{x+1}{B}{}^{\kappa}_{r_{x+1}}K_{\nu\mu\lambda\kappa}-2D_{s_0}\overset{x+1}{H}_{r_0]q_x r_{x+1}})=0.
\end{aligned}
\tag{5.8}
$$

At this point we note that the Codazzi equations $\overset{x-1}{N}$, $\overset{x+1}{N}$, $\overset{x}{N}$, $\overset{x}{N}$ are suggested by the first, second, third, and fourth terms of (5.8) respectively. If we assume that the Codazzi equation $\overset{x-1}{N}$ holds, then

[33] II, pg. 84.

[33] See (3.8) and I, pg. 124.

[34] The existence of such an identity and its application to this case will be demonstrated in Lemma V.1.

it will also be true that $\overset{x}{N}_{x-1}$ holds,[35] and if we change the form of the fourth term of (5.8) by using the Third Identity for the Riemann-Christoffel affinor,[36] then we shall be able to write (5.8) as

$$\begin{aligned}(5.9)\quad & \overset{x+1}{H}_{[u_0|q_x|}{}^{\cdot\cdot r_{x+1}}(\overset{0}{B}{}^{p_x}_{s_0r_0]}\overset{x+1}{B}{}^{\lambda}_{r_{x+1}}\overset{x}{B}{}^{\kappa}_{p_x}K_{p\lambda\kappa} + 2D_{s_0}\overset{x}{H}_{r_0]p_xr_{x+1}}) \\ & - \overset{x+1}{H}_{[u_0|p_x|}{}^{\cdot\cdot r_{x+1}}(\overset{0}{B}{}^{p_x}_{s_0r_0]}\overset{x+1}{B}{}^{\lambda}_{r_{x+1}}\overset{x}{B}{}^{\kappa}_{q_x}K_{p\lambda\kappa} + 2D_{s_0}\overset{x}{H}_{r_0|q_xr_{x+1}}) = 0.\end{aligned}$$

Now our course is clear. We define $\theta_{s_0r_0r_{x+1}p_x}$ by means of

$$(5.10)\quad \theta_{s_0r_0r_{x+1}p_x} = \overset{0}{B}{}^{p_x}_{s_0r_0}\overset{x+1}{B}{}^{\lambda}_{r_{x+1}}\overset{x}{B}{}^{\kappa}_{p_x}K_{p\lambda\kappa} + 2D_{[s_0}\overset{x+1}{H}_{r_0]p_xr_{x+1}},$$

and use the First Identity for the Riemann-Christoffel affinor[37] to show that

$$(5.11)\quad \theta_{[s_0r_0]r_{x+1}p_x} = \theta_{s_0r_0r_{x+1}p_x}.$$

Then we rewrite (5.9) as

$$(5.12)\quad \overset{x+1}{H}_{[u_0|q_x|}{}^{\cdot\cdot r_{x+1}}\theta_{s_0r_0]r_{x+1}p_x} - \overset{x+1}{H}_{[u_0|p_x|}{}^{\cdot\cdot r_{x+1}}\theta_{s_0r_0]r_{x+1}q_x} = 0,$$

or as

$$\begin{aligned}(5.13)\quad & \overset{x+1}{H}_{u_0q_x}{}^{\cdot\cdot r_{x+1}}\theta_{s_0r_0r_{x+1}p_x} + \overset{x+1}{H}_{s_0q_x}{}^{\cdot\cdot r_{x+1}}\theta_{r_0u_0r_{x+1}p_x} + \overset{x+1}{H}_{r_0q_x}{}^{\cdot\cdot r_{x+1}}\theta_{u_0s_0r_{x+1}p_x} \\ & - \overset{x+1}{H}_{u_0p_x}{}^{\cdot\cdot r_{x+1}}\theta_{s_0r_0r_{x+1}q_x} - \overset{x+1}{H}_{s_0p_x}{}^{\cdot\cdot r_{x+1}}\theta_{r_0u_0r_{x+1}q_x} \\ & - \overset{x+1}{H}_{r_0p_x}{}^{\cdot\cdot r_{x+1}}\theta_{u_0s_0r_{x+1}q_x} = 0.\end{aligned}$$

If we can show that $\theta_{s_0r_0r_{x+1}p_x}$ vanishes as a consequence of (5.13), then we will have shown that the Codazzi equation $\overset{x+1}{N}_{x}$ is a consequence of the Gauss equation $\overset{x}{N}_{x}$ and the Codazzi equation $\overset{x}{N}_{x-1}$. We shall see under what conditions $\theta_{s_0r_0r_{x+1}p_x}$ vanishes.

Let us assume first that condition 0 for $\overset{x+1}{H}$ is satisfied so that we can use the vectors $\overset{x+1}{u}{}^{r_0}_{A}$, $\overset{x+1}{v}{}^{r_0}_{B}$, $\overset{x+1}{z}{}^{p_x}_{B}$, $\overset{x+1}{y}{}^{p_x}_{C}$ thus defined.[38] Transvect in

[35] See (5.1), (5.3).
[36] See (5.2).
[37] See (3.16) and I, pg. 113.
[38] See (2.1), (2.2), and (2.3).

(5.13) by $\overset{x+1}{\underset{A_1}{u^{s_0}}}\ \overset{x+1}{\underset{A_2}{u^{r_0}}}\ \overset{x+1}{\underset{C}{y^{p_x}}}$. We have

$$\overset{x+1}{H}_{u_0 q_x \cdots r_{x+1}}\ \overset{x+1}{\underset{A_1}{u^{s_0}}}\ \overset{x+1}{\underset{A_2}{u^{r_0}}}\ \overset{x+1}{\underset{C}{y^{p_x}}}\theta_{s_0 r_0 r_{x+1} p_x} = 0, \tag{5.14}$$

where A_1, A_2, and C can be chosen arbitrarily. We can consider A_1, A_2, C fixed once arbitrarily chosen, and then (5.14) tells us that the vector $\overset{x+1}{\underset{A_1}{u^{s_0}}}\ \overset{x+1}{\underset{A_2}{u^{r_0}}}\ \overset{x+1}{\underset{C}{y^{p_x}}}\theta_{s_0 r_0 r_{x+1} p_x}$, which lies in the $(x+1)^{st}$ normal space, is normal to the m independent vectors included in the set of vectors $\overset{x+1}{\underset{B}{v^{u_0}}}\ \overset{x+1}{\underset{B}{z^{q_x}}}\ \overset{x+1}{H}_{u_0 q_x \cdots r_{x+1}}$, all of which also lie in the $(x+1)^{st}$ normal space.[39] Hence we conclude that

$$\overset{x+1}{\underset{A_1}{u^{s_0}}}\ \overset{x+1}{\underset{A_2}{u^{r_0}}}\ \overset{x+1}{\underset{C}{y^{p_x}}}\theta_{s_0 r_0 r_{x+1} p_x} = 0 \tag{5.15}$$

where A_1, A_2, and C can be chosen arbitrarily.

Now we transvect in (5.13) by $\overset{x+1}{\underset{A_1}{u^{s_0}}}\ \overset{x+1}{\underset{A_2}{u^{r_0}}}\ \overset{x+1}{\underset{B_1}{z^{p_x}}}\ \overset{x+1}{\underset{B_2}{v^{u_0}}}$. We have

$$\overset{x+1}{\underset{B_2}{v^{u_0}}}\ \overset{x+1}{H}_{u_0 q_x \cdots r_{x+1}}\ \overset{x+1}{\underset{A_1}{u^{s_0}}}\ \overset{x+1}{\underset{A_2}{u^{r_0}}}\ \overset{x+1}{\underset{B_1}{z^{p_x}}}\theta_{s_0 r_0 r_{x+1} p_x} = 0, \tag{5.16}$$

where A_1, A_2, and B_1 can be chosen arbitrarily, but where $B_2 \neq B_1$. If we assume that condition 1 is satisfied for $\overset{x+1}{H}$, and if A_1, A_2, and B_1 be considered fixed once arbitrarily chosen, then we see from (5.16) that the vector $\overset{x+1}{\underset{A_1}{u^{s_0}}}\ \overset{x+1}{\underset{A_2}{u^{r_0}}}\ \overset{x+1}{\underset{B_1}{z^{p_x}}}\theta_{s_0 r_0 r_{x+1} p_x}$ is normal to m other independent vectors in the $(x+1)^{st}$ normal space, namely the m independent vectors included in the set $\overset{x+1}{\underset{B}{v^{u_0}}}\ \overset{x+1}{\underset{B}{z^{q_x}}}\ \overset{x+1}{H}_{u_0 q_x \cdots r_{x+1}}$, $B \neq B_1$. Hence,

$$\overset{x+1}{\underset{A_1}{u^{s_0}}}\ \overset{x+1}{\underset{A_2}{u^{r_0}}}\ \overset{x+1}{\underset{B_1}{z^{p_x}}}\theta_{s_0 r_0 r_{x+1} p_x} = 0, \tag{5.17}$$

where A_1, A_2, and B_1 can be chosen arbitrarily.

We transvect now in (5.13) by $\overset{x+1}{\underset{B_1}{v^{s_0}}}\ \overset{x+1}{\underset{A_1}{u^{r_0}}}\ \overset{x+1}{\underset{C}{y^{p_x}}}\ \overset{x+1}{\underset{B_2}{z^{q_x}}}$, and we can then

[39] See Lemma II.1.

write

$$\overset{x+1}{z}{}^{q_x}_{B_2}\, \overset{x+1}{H}{}_{u_0 q_x}^{\cdots r_{x+1}}\, \overset{x+1}{v}{}^{s_0}_{B_1}\, \overset{x+1}{u}{}^{r_0}_{A_1}\, \overset{x+1}{y}{}^{p_x}_{C}\, \theta_{s_0 r_0 r_{x+1} p_x} = 0, \tag{5.18}$$

where A_1, C, and B_1 can be chosen arbitrarily, but where $B_2 \neq B_1$. Just as in the consideration of (5.16), we can conclude from (5.18) that

$$\overset{x+1}{v}{}^{s_0}_{B_1}\, \overset{x+1}{u}{}^{r_0}_{A_1}\, \overset{x+1}{y}{}^{p_x}_{C}\, \theta_{s_0 r_0 r_{x+1} p_x} = 0, \tag{5.19}$$

where A_1, B_1, and C can be chosen arbitrarily, if condition 1 for $\overset{x+1}{H}$ is satisfied. (5.19) and (5.11) together tell us that under the same assumption,

$$\overset{x+1}{u}{}^{s_0}_{A_1}\, \overset{x+1}{v}{}^{r_0}_{B_1}\, \overset{x+1}{y}{}^{p_x}_{C}\, \theta_{s_0 r_0 r_{x+1} p_x} = 0. \tag{5.20}$$

We transvect this time in (5.13) by $\overset{x+1}{v}{}^{s_0}_{B_1}\, \overset{x+1}{u}{}^{r_0}_{A_1}\, \overset{x+1}{z}{}^{p_x}_{B_2}\, \overset{x+1}{v}{}^{u_0}_{B_3}\, \overset{x+1}{z}{}^{q_x}_{B_3}$.

$$\overset{x+1}{v}{}^{u_0}_{B_3}\, \overset{x+1}{z}{}^{q_x}_{B_3}\, \overset{x+1}{H}{}_{u_0 q_x}^{\cdots r_{x+1}}\, \overset{x+1}{v}{}^{s_0}_{B_1}\, \overset{x+1}{u}{}^{r_0}_{A_1}\, \overset{x+1}{z}{}^{p_x}_{B_2}\, \theta_{s_0 r_0 r_{x+1} p_x} = 0, \tag{5.21}$$

where A_1 and B_1 can be chosen arbitrarily, but where $B_2 \neq B_1$, $B_3 \neq B_1$, B_2. Here we can conclude that if condition 2 for $\overset{x+1}{H}$ is satisfied, and if B_1, A_1, and B_2 be considered fixed once chosen, then the vector $\overset{x+1}{v}{}^{s_0}_{B_1}\, \overset{x+1}{u}{}^{r_0}_{A_1}\, \overset{x+1}{z}{}^{p_x}_{B_2}\, \theta_{s_0 r_0 r_{x+1} p_x}$ is normal to the m independent vectors of the $\overset{x+1}{m}$ $(x+1)^{st}$ normal space which are included in the set of vectors $\overset{x+1}{v}{}^{u_0}_{B}\, \overset{x+1}{z}{}^{q_x}_{B}\, \overset{x+1}{H}{}_{u_0 q_x}^{\cdots r_{x+1}}$, $B \neq B_1, B_2$. Hence it is true that

$$\overset{x+1}{v}{}^{s_0}_{B_1}\, \overset{x+1}{u}{}^{r_0}_{A_1}\, \overset{x+1}{z}{}^{p_x}_{B_2}\, \theta_{s_0 r_0 r_{x+1} p_x} = 0, \tag{5.22}$$

where A_1 and B_1 can be chosen arbitrarily, but where $B_2 \neq B_1$. (5.11) enables us to add the following statement, true under the same conditions:

$$\overset{x+1}{u}{}^{s_0}_{A_1}\, \overset{x+1}{v}{}^{r_0}_{B_1}\, \overset{x+1}{z}{}^{p_x}_{B_2}\, \theta_{s_0 r_0 r_{x+1} p_x} = 0. \tag{5.23}$$

Transvection in (5.13) by $\overset{x+1}{v}{}^{s_0}_{B_1}\, \overset{x+1}{v}{}^{r_0}_{B_2}\, \overset{x+1}{y}{}^{p_x}_{C}\, \overset{x+1}{z}{}^{q_x}_{B_3}$ yields

$$(5.24)\qquad \overset{x+1}{\underset{B_3}{z^{q_x}}}\ \overset{x+1}{H_{u_0 q_x \cdots r_{x+1}}}\ \overset{x+1}{\underset{B_1}{v^{s_0}}}\ \overset{x+1}{\underset{B_2}{v^{r_0}}}\ \overset{x+1}{\underset{C}{y^{p_x}}}\theta_{s_0 r_0 r_{x+1} p_x} = 0,$$

where B_1, B_2, and C can be chosen arbitrarily, but where $B_3 \neq B_1, B_2$. Again we make use of the fact that when condition 2 for $\overset{x+1}{H}$ is satisfied, $\underset{x+1}{m}$ independent vectors are included in the set $\overset{x+1}{\underset{B}{v^{u_0}}}\ \overset{x+1}{\underset{B}{z^{q_x}}}\ \overset{x+1}{H_{u_0 q_x \cdots r_{x+1}}}$; $B \neq B_1, B_2$. But (5.24) means that the vector $\overset{x+1}{\underset{B_1}{v^{s_0}}}\ \overset{x+1}{\underset{B_2}{v^{r_0}}}\ \overset{x+1}{\underset{C}{y^{p_x}}}\theta_{s_0 r_0 r_{x+1} p_x}$ of the $(x+1)^{st}$ normal space, (B_1, B_2, C considered fixed once arbitrarily chosen), is normal to these $\underset{x+1}{m}$ independent vectors, and it therefore vanishes.

$$(5.25)\qquad \overset{x+1}{\underset{B_1}{v^{s_0}}}\ \overset{x+1}{\underset{B_2}{v^{r_0}}}\ \overset{x+1}{\underset{C}{y^{p_x}}}\theta_{s_0 r_0 r_{x+1} p_x} = 0,$$

where B_1, B_2, and C can be chosen arbitrarily.

If we transvect in (5.13) by $\overset{x+1}{\underset{B_1}{v^{s_0}}}\ \overset{x+1}{\underset{B_2}{v^{r_0}}}\ \overset{x+1}{\underset{B_3}{z^{p_x}}}\ \overset{x+1}{\underset{B_4}{z^{q_x}}}\ \overset{x+1}{\underset{B_4}{v^{u_0}}}$, we have

$$(5.26)\qquad \overset{x+1}{\underset{B_4}{v^{u_0}}}\ \overset{x+1}{\underset{B_4}{z^{q_x}}}\ \overset{x+1}{H_{u_0 q_x \cdots r_{x+1}}}\ \overset{x+1}{\underset{B_1}{v^{s_0}}}\ \overset{x+1}{\underset{B_2}{v^{r_0}}}\ \overset{x+1}{\underset{B_3}{z^{p_x}}}\theta_{s_0 r_0 r_{x+1} p_x} = 0,$$

where B_1 and B_2 can be chosen arbitrarily, but where $B_3 \neq B_1, B_2$ and $B_4 \neq B_1, B_2, B_3$. Suppose that condition 3 for $\overset{x+1}{H}$ were satisfied. Then there would be $\underset{x+1}{m}$ independent vectors in the set $\overset{x+1}{\underset{B}{v^{u_0}}}\ \overset{x+1}{\underset{B}{z^{q_x}}}\ \overset{x+1}{H_{u_0 q_x \cdots r_{x+1}}}$, $B \neq B_1, B_2, B_3$, all lying in the $(x+1)^{st}$ normal space. The vector $\overset{x+1}{\underset{B_1}{v^{s_0}}}\ \overset{x+1}{\underset{B_2}{v^{r_0}}}\ \overset{x+1}{\underset{B_3}{z^{p_x}}}\theta_{s_0 r_0 r_{x+1} p_x}$ (B_1, B_2, B_3 considered fixed once chosen), which also lies in the $(x+1)^{st}$ normal space, would vanish because it would be normal to the $\underset{x+1}{m}$ independent vectors of the above mentioned set by (5.26). Thus, if condition 3 for $\overset{x+1}{H}$ is satisfied,

$$(5.27)\qquad \overset{x+1}{\underset{B_1}{v^{s_0}}}\ \overset{x+1}{\underset{B_2}{v^{r_0}}}\ \overset{x+1}{\underset{B_3}{z^{p_x}}}\theta_{s_0 r_0 r_{x+1} p_x} = 0,$$

where B_1 and B_2 can be chosen arbitrarily, but where $B_3 \neq B_1, B_2$.

In addition to (5.15), (5.17), (5.19), (5.20), (5.22), (5.23), (5.25), and (5.27) we need statements which would tell us that $\overset{x+1}{\underset{B_1}{v^{s_0}}}\ \overset{x+1}{\underset{A_1}{u^{r_0}}}\ \overset{x+1}{\underset{B_1}{z^{p_x}}}\theta_{s_0 r_0 r_{x+1} p_x}$

and $\overset{s+1}{\underset{B_1}{v^{s_0}}}\ \overset{s+1}{\underset{B_2}{v^{r_0}}}\ \overset{s+1}{\underset{B_1}{z^{p_x}}}\theta_{s_0r_0r_{s+1}p_x}$ vanish. Then (5.11) would tell us that $\overset{s+1}{\underset{A_1}{u^{s_0}}}\ \overset{s+1}{\underset{B_1}{v^{r_0}}}\ \overset{s+1}{\underset{B_1}{z^{p_x}}}\theta_{s_0r_0r_{s+1}p_x}$, $\overset{s+1}{\underset{B_2}{v^{s_0}}}\ \overset{s+1}{\underset{B_1}{v^{r_0}}}\ \overset{s+1}{\underset{B_1}{z^{p_x}}}\theta_{s_0r_0r_{s+1}p_x}$, and $\overset{s+1}{\underset{B_1}{v^{s_0}}}\ \overset{s+1}{\underset{B_1}{v^{r_0}}}\ \overset{s+1}{\underset{B_1}{z^{p_x}}}\theta_{s_0r_0r_{s+1}p_x}$ also vanish, and we could conclude that $\theta_{s_0r_0r_{s+1}p_x}$ vanishes, which is our goal. To show that $\overset{s+1}{\underset{B_1}{v^{s_0}}}\ \overset{s+1}{\underset{A_1}{u^{r_0}}}\ \overset{s+1}{\underset{B_1}{z^{p_x}}}\theta_{s_0r_0r_{s+1}p_x}$ and $\overset{s+1}{\underset{B_1}{v^{s_0}}}\ \overset{s+1}{\underset{B_2}{v^{r_0}}}\ \overset{s+1}{\underset{B_1}{z^{p_x}}}\theta_{s_0r_0r_{s+1}p_x}$ vanish, we shall have to use an argument different from those we have used before, and we shall have to make additional assumptions.

Transvect in (5.13) by $\overset{s+1}{\underset{B_1}{v^{s_0}}}\ \overset{s+1}{\underset{A_1}{u^{r_0}}}\ \overset{s+1}{\underset{B_1}{z^{p_x}}}\ \overset{s+1}{\underset{B_2}{z^{q_x}}}\ \overset{s+1}{\underset{B_2}{v^{u_0}}}$.

$$\overset{s+1}{\underset{B_2}{v^{u_0}}}\ \overset{s+1}{\underset{B_2}{z^{q_x}}}\ \overset{s+1}{H}_{u_0q_x}^{\cdot\cdot\cdot r_{s+1}}\ \overset{s+1}{\underset{B_1}{v^{s_0}}}\ \overset{s+1}{\underset{A_1}{u^{r_0}}}\ \overset{s+1}{\underset{B_1}{z^{p_x}}}\theta_{s_0r_0r_{s+1}p_x} - \overset{s+1}{\underset{B_1}{v^{s_0}}}\ \overset{s+1}{\underset{B_1}{z^{p_x}}}\ \overset{s+1}{H}_{s_0p_x}^{\cdot\cdot\cdot r_{s+1}}\ \overset{s+1}{\underset{A_1}{u^{r_0}}}\ \overset{s+1}{\underset{B_2}{v^{u_0}}}\ \overset{s+1}{\underset{B_2}{z^{q_x}}}\theta_{r_0u_0r_{s+1}q_x} = 0, \tag{5.28}$$

where A_1 and B_1 can be chosen arbitrarily, but where $B_2 \neq B_1$; or, using (5.11) and relettering dummy indices,

$$\overset{s+1}{\underset{B_2}{v^{u_0}}}\ \overset{s+1}{\underset{B_2}{z^{q_x}}}\ \overset{s+1}{H}_{u_0q_x}^{\cdot\cdot\cdot r_{s+1}}\ \overset{s+1}{\underset{B_1}{v^{s_0}}}\ \overset{s+1}{\underset{A_1}{u^{r_0}}}\ \overset{s+1}{\underset{B_1}{z^{p_x}}}\theta_{s_0r_0r_{s+1}p_x} + \overset{s+1}{\underset{B_1}{v^{u_0}}}\ \overset{s+1}{\underset{B_1}{z^{q_x}}}\ \overset{s+1}{H}_{u_0q_x}^{\cdot\cdot\cdot r_{s+1}}\ \overset{s+1}{\underset{B_2}{v^{s_0}}}\ \overset{s+1}{\underset{A_1}{u^{r_0}}}\ \overset{s+1}{\underset{B_2}{z^{p_x}}}\theta_{s_0r_0r_{s+1}p_x} = 0. \tag{5.29}$$

To make our discussion of (5.29) formally simpler, let us write

$$\underset{B}{H^{r_{s+1}}} = \overset{s+1}{\underset{B}{v^{u_0}}}\ \overset{s+1}{\underset{B}{z^{q_x}}}\ \overset{s+1}{H}_{u_0q_x}^{\cdot\cdot\cdot r_{s+1}}; \qquad \underset{B}{\theta_{r_{s+1}}} = \overset{s+1}{\underset{B}{v^{s_0}}}\ \overset{s+1}{\underset{A_1}{u^{r_0}}}\ \overset{s+1}{\underset{B}{z^{p_x}}}\theta_{s_0r_0r_{s+1}p_x}. \tag{5.30}$$

Then we can rewrite (5.29) as

$$\underset{B_2}{H^{r_{s+1}}}\underset{B_1}{\theta_{r_{s+1}}} + \underset{B_1}{H^{r_{s+1}}}\underset{B_2}{\theta_{r_{s+1}}} = 0, \tag{5.31}$$

where B_1 can be chosen arbitrarily, but where $B_2 \neq B_1$.

(5.31) can be considered as a set of linear equations. There are $\underset{s+1}{\beta}\ \underset{s+1}{m}$ unknowns, $\underset{B}{\theta_{r_{s+1}}}$, and $\overset{\beta}{\underset{s+1}{C_2}}$ equations. Hence a necessary condition for the vanishing of the vectors $\underset{B}{\theta_{r_{s+1}}}$ is $\underset{s+1}{\beta}\ \underset{s+1}{m} \leq \overset{\beta}{\underset{s+1}{C_2}} = \dfrac{\beta\,(\beta-1)}{2}$; $\underset{s+1}{m} \leq \dfrac{\beta-1}{2}$. We cannot be sure, however, that this condition is

sufficient, for even if it should be satisfied, we would not know yet whether there are $\underset{s+1}{\beta}\ \underset{s+1}{m}$ independent equations in the set (5.31).

For the case $\underset{s+1}{m} = 1$, $\underset{s+1}{\beta} \geq 3$, there are $\underset{s+1}{\beta}$ independent equations in the set (5.31). Suppose that $\underset{s+1}{\beta} = 3$, and write the equations (5.31) as

$$\begin{aligned} &1.\quad \underset{1}{H^{r_{s+1}}}\underset{2}{\theta_{r_{s+1}}} + \underset{2}{H^{r_{s+1}}}\underset{1}{\theta_{r_{s+1}}} = 0 \\ &2.\quad \underset{3}{H^{r_{s+1}}}\underset{1}{\theta_{r_{s+1}}} + \underset{1}{H^{r_{s+1}}}\underset{3}{\theta_{r_{s+1}}} = 0 \\ &3.\quad \underset{2}{H^{r_{s+1}}}\underset{3}{\theta_{r_{s+1}}} + \underset{3}{H^{r_{s+1}}}\underset{2}{\theta_{r_{s+1}}} = 0. \end{aligned} \tag{5.32}$$

There is only one independent vector among $\underset{B}{H^{r_{s+1}}}$ because $\underset{s+1}{m} = 1$.[40] Because none of the vectors $\underset{B}{H^{r_{s+1}}}$ are null vectors,[41] we can say that

$$\underset{2}{H^{r_{s+1}}} = A\underset{1}{H^{r_{s+1}}}, \qquad \underset{3}{H^{r_{s+1}}} = B\underset{1}{H^{r_{s+1}}}, \qquad AB \neq 0, \tag{5.33}$$

and rewrite (5.32) as

$$\begin{aligned} &1.\quad \underset{1}{H^{r_{s+1}}}(\underset{2}{\theta_{r_{s+1}}} + A\underset{1}{\theta_{r_{s+1}}}) = 0 \\ &2.\quad \underset{1}{H^{r_{s+1}}}(B\underset{1}{\theta_{r_{s+1}}} + \underset{3}{\theta_{r_{s+1}}}) = 0 \\ &3.\quad \underset{1}{H^{r_{s+1}}}(A\underset{3}{\theta_{r_{s+1}}} + B\underset{2}{\theta_{r_{s+1}}}) = 0. \end{aligned} \tag{5.34}$$

Because any vector in a one dimensional space which is normal to a non-null vector must be itself a null vector, we have from (5.34)

$$\begin{aligned} &1.\quad A\underset{1}{\theta_{r_{s+1}}} + \underset{2}{\theta_{r_{s+1}}} = 0 \\ &2.\quad B\underset{1}{\theta_{r_{s+1}}} + \underset{3}{\theta_{r_{s+1}}} = 0 \\ &3.\quad B\underset{2}{\theta_{r_{s+1}}} + A\underset{3}{\theta_{r_{s+1}}} = 0. \end{aligned} \tag{5.35}$$

But here, because the determinant of the coefficients $= -2AB$ and cannot vanish because of (5.33), we must conclude that $\underset{1}{\theta_{r_{s+1}}}$, $\underset{2}{\theta_{r_{s+1}}}$, $\underset{3}{\theta_{r_{s+1}}}$ vanish. We can write this result as

[40] See Lemma II.1.
[41] See (2.2).

$$(5.36)\qquad \overset{x+1}{B}\ \overset{x+1}{A_1}\ \overset{x+1}{B}\ v^{s_0}\ u^{r_0}\ z^{p_x}\theta_{s_0r_0r_{x+1}p_x} = -\overset{x+1}{A_1}\ \overset{x+1}{B}\ \overset{x+1}{B}\ u^{s_0}\ v^{r_0}\ z^{p_x}\theta_{s_0r_0r_{x+1}p_x} = 0,$$

where A_1 and B can be chosen arbitrarily, if condition 0 for H^{x+1} is satisfied, $m_{x+1} = 1$, and $\beta_{x+1} = 3$.

It is not hard to see that this result is true more generally for $\beta_{x+1} \geq 3$. In that case we can consider the indices B three at a time, using those equations of (5.31) which concern these three indices and only these three. Then the work we have just done applies, and $\overset{x+1}{B}\ \overset{x+1}{A_1}\ \overset{x+1}{B}\ v^{s_0}\ u^{r_0}\ z^{p_x}\theta_{s_0r_0r_{x+1}p_x}$ vanishes for the three particular values of B chosen.[42]

If one considers (5.31) for the special case $m_{x+1} = 2$, one finds that there are not enough independent equations in (5.31) to enable one to conclude without further information that $\overset{x+1}{B}\ \overset{x+1}{A_1}\ \overset{x+1}{B}\ v^{s_0}\ u^{r_0}\ z^{p_x}\theta_{s_0r_0r_{x+1}p_x}$ vanishes.[43] Therefore we shall add to the assumptions we have already made the assumption that $m_{x+1} = 1$.

Now we shall consider $\overset{x+1}{B_1}\ \overset{x+1}{B_2}\ \overset{x+1}{B_1}\ v^{s_0}\ v^{r_0}\ z^{p_x}\theta_{s_0r_0r_{x+1}p_x}$. Transvect in (5.13) by

$$\overset{x+1}{B_1}\ \overset{x+1}{B_2}\ \overset{x+1}{B_1}\ \overset{x+1}{B_3}\ \overset{x+1}{B_3}\ v^{s_0}\ v^{r_0}\ z^{p_x}\ v^{u_0}\ z^{q_x}.$$

$$(5.37)\qquad \begin{aligned} &\overset{x+1}{B_3}\ \overset{x+1}{B_3}\ H^{\cdots r_{x+1}}_{u_0q_x}\ \overset{x+1}{B_1}\ \overset{x+1}{B_2}\ \overset{x+1}{B_1}\ v^{s_0}\ v^{r_0}\ z^{p_x}\theta_{s_0r_0r_{x+1}p_x} \\ &\quad - \overset{x+1}{B_1}\ \overset{x+1}{B_1}\ H^{\cdots r_{x+1}}_{s_0p_x}\ \overset{x+1}{B_2}\ \overset{x+1}{B_3}\ \overset{x+1}{B_3}\ v^{r_0}\ v^{u_0}\ z^{q_x}\theta_{r_0u_0r_{x+1}q_x} = 0, \end{aligned}$$

where B_1 can be chosen arbitrarily, but where $B_2 \neq B_1$; $B_3 \neq B_1$, B_2. We use (5.11) and reletter some dummy indices to write (5.37) as

$$(5.38)\qquad \begin{aligned} &\overset{x+1}{B_3}\ \overset{x+1}{B_3}\ H^{\cdots r_{x+1}}_{u_0q_x}\ \overset{x+1}{B_1}\ \overset{x+1}{B_2}\ \overset{x+1}{B_1}\ v^{s_0}\ v^{r_0}\ z^{p_x}\theta_{s_0r_0r_{x+1}p_x} \\ &\quad + \overset{x+1}{B_1}\ \overset{x+1}{B_1}\ H^{\cdots r_{x+1}}_{u_0q_x}\ \overset{x+1}{B_3}\ \overset{x+1}{B_2}\ \overset{x+1}{B_3}\ v^{s_0}\ v^{r_0}\ z^{p_x}\theta_{s_0r_0r_{x+1}p_x} = 0. \end{aligned}$$

[42] The demonstration used for this special case is unnecessarily long. However, it is useful for comparison with some work below. See V A 1.

[43] See V A 1.

We consider (5.38) in almost exactly the same way we considered (5.29). We have $\underset{s+1}{m} = 1$, and we define

$$(5.39)\quad \underset{B}{H^{r_{s+1}}} = \overset{s+1}{\underset{B}{v^{u_0}}}\ \overset{s+1}{\underset{B}{z^{q_s}}}\ H_{u_0 q_s \cdots r_{s+1}}; \qquad \underset{B_1 B_2}{\theta_{r_{s+1}}} = \overset{s+1}{\underset{B_1}{v^{s_0}}}\ \overset{s+1}{\underset{B_2}{v^{r_0}}}\ \overset{s+1}{\underset{B_1}{z^{p_s}}}\theta_{s_0 r_0 r_{s+1} p_s}.$$

These definitions enable us to rewrite (5.38) in the form

$$(5.40)\quad \underset{B_3}{H^{r_{s+1}}}\ \underset{B_1 B_2}{\theta_{r_{s+1}}} + \underset{B_1}{H^{r_{s+1}}}\ \underset{B_3 B_2}{\theta_{r_{s+1}}} = 0,$$

where B_2 can be chosen arbitrarily, but where $B_1 \neq B_2$; $B_3 \neq B_1, B_2$.

If B_2 be chosen arbitrarily and then considered fixed, the set of equations (5.40) is a set of $C_2^{\underset{s+1}{\beta}-1} = \dfrac{(\underset{s+1}{\beta}-1)(\underset{s+1}{\beta}-2)}{2}$ linear equations in the $\underset{s+1}{\beta} - 1$ unknowns $\underset{B_1 B_2}{\theta_{r_{s+1}}}$. Hence, a necessary condition that $\underset{B_1 B_2}{\theta_{r_{s+1}}}$ vanish as a consequence of (5.40) is that $1 \leq \dfrac{\underset{s+1}{\beta}-2}{2}$; $\underset{s+1}{\beta} \geq 4$. This condition is also sufficient, for we can show exactly as for (5.31) that the equations (5.40) contain $\underset{s+1}{\beta} - 1$ independent equations among them. Suppose that $\underset{s+1}{\beta} = 4$. We can without loss of generality, take B_2 as 4, and then write our equations (5.40) as

$$(5.41)\quad \begin{aligned} &1.\ \underset{1}{H^{r_{s+1}}}\ \underset{24}{\theta_{r_{s+1}}} + \underset{2}{H^{r_{s+1}}}\ \underset{14}{\theta_{r_{s+1}}} = 0 \\ &2.\ \underset{3}{H^{r_{s+1}}}\ \underset{14}{\theta_{r_{s+1}}} + \underset{1}{H^{r_{s+1}}}\ \underset{34}{\theta_{r_{s+1}}} = 0 \\ &3.\ \underset{2}{H^{r_{s+1}}}\ \underset{34}{\theta_{r_{s+1}}} + \underset{3}{H^{r_{s+1}}}\ \underset{24}{\theta_{r_{s+1}}} = 0. \end{aligned}$$

But (5.41) and (5.32) are the same system of equations, and one can therefore conclude that $\underset{B_4}{\theta_{r_{s+1}}}$ vanishes. However, B_2 could have been chosen as any index, and so we see that $\underset{B_1 B_2}{\theta_{r_{s+1}}}$ vanishes if $B_1 \neq B_2$. We can write this result in the desired form:

$$(5.42)\quad \overset{s+1}{\underset{B_1}{v^{s_0}}}\ \overset{s+1}{\underset{B_2}{v^{r_0}}}\ \overset{s+1}{\underset{B_1}{z^{p_s}}}\theta_{s_0 r_0 r_{s+1} p_s} = 0,$$

where B_1 can be chosen arbitrarily, but where $B_2 \neq B_1$, if $\underset{s+1}{m} = 1$, $\underset{s+1}{\beta} = 4$.

If $\underset{x+1}{\beta} \geq 4$, (5.42) also holds because then we can consider our indices four at a time and draw our conclusions from those equations of the set (5.40) that contain these four selected indices and only these four.

We are now in a position to summarize the work of this section. Equations (5.15), (5.17), (5.19), (5.20), (5.22), (5.23), (5.25), (5.27), (5.36), and (5.42) enable us to conclude that $\theta_{s_0 r_0 r_{x+1} p_x}$ vanishes. But this means that the Codazzi equation $\underset{x}{\overset{x+1}{N}}$ holds, and therefore we can state

THEOREM V. If $\underset{x+1}{m} = 1$, and if condition 3 for $\overset{x+1}{H}$ is satisfied, $x = 0, \cdots, k$, then the Codazzi equation $\underset{x}{\overset{x+1}{N}}$ is a consequence of the Gauss equation $\underset{x}{\overset{x}{N}}$ and the Codazzi equation $\underset{x-1}{\overset{x}{N}}$; or, symbolically stated, $\underset{x}{\overset{x}{N}}, \underset{x-1}{\overset{x}{N}} \rightarrow \underset{x}{\overset{x+1}{N}}$.

In stating Theorem V we have used the fact that when $\underset{x+1}{m} = 1$, the statements "$\underset{x}{\beta} \geq A + 1$" and "Condition A for $\overset{x}{H}$ is satisfied" are equivalent statements.[44]

In arriving at (5.8) we assumed that the following identity for $\overset{x}{K}_{s_0 r_0 q_x p_x}$, analagous to the Bianchi Identity for the Riemann-Christoffel affinor, exists:

$$D_{[u_0} \overset{x}{K}_{s_0 r_0] q_x p_x} = 0. \tag{5.43}$$

[45]

Therefore, to complete our theorem, we must prove the

LEMMA V.1. $$D_{[u_0} \overset{x}{K}_{s_0 r_0] q_x p_x} = 0.$$

We shall prove this lemma just as Schouten and Struik demonstrate Bianchi's Identity for the Riemann-Christoffel affinor.[46] The affinor $\overset{x}{K}_{s_0 r_0 q_x p_x}$ is defined by

$$D_{[s_0} D_{r_0]} w_{q_x} = -\tfrac{1}{2} \overset{x}{K}_{s_0 r_0 q_x}{}^{\cdots p_x} w_{p_x}. \tag{5.44} \equiv (1.17)$$

[44] See Lemma II.2.

[45] See discussion following (5.7).

[46] I, pgs. 123, 124.

Applying (5.44) to the affinor $D_{u_0} w_{q_x}$, we have

$$(5.45)\quad D_{[s_0} D_{r_0]}(D_{u_0} w_{q_x}) = -\tfrac{1}{2} \overset{0}{K}{}^{\cdot\cdot\cdot\cdot t_0}_{s_0 r_0 u_0} D_{t_0} w_{q_x} - \tfrac{1}{2} \overset{x}{K}{}^{\cdot\cdot\cdot\cdot p_x}_{s_0 r_0 q_x} D_{u_0} w_{p_x}.$$

If we alternate on the indices s_0, r_0, u_0, we can write

$$(5.46)\quad D_{[s_0} D_{r_0} D_{u_0]} w_{q_x} = -\tfrac{1}{2} \overset{0}{K}{}^{\cdot\cdot\cdot\cdot t_0}_{[s_0 r_0 u_0]} D_{t_0} w_{q_x} - \tfrac{1}{2} \overset{x}{K}{}^{\cdot\cdot\cdot\cdot p_x}_{[s_0 r_0 | q_x |}\, D_{u_0]} w_{p_x},$$

which becomes

$$(5.47)\quad D_{[s_0} D_{r_0} D_{u_0]} w_{q_x} = -\tfrac{1}{2} \overset{x}{K}{}^{\cdot\cdot\cdot\cdot p_x}_{[s_0 r_0 | q_x |}\, D_{u_0]} w_{p_x}$$

when we use the Second Identity for the Riemann-Christoffel affinor $\overset{0}{K}$,

$$(5.48)\quad \overset{0}{K}{}^{\cdot\cdot\cdot\cdot t_0}_{[s_0 r_0 u_0]} = 0.^{47}$$

But we can arrive at another expression for $D_{[s_0} D_{r_0} D_{u_0]} w_{q_x}$. We can differentiate (5.44) and then alternate on the indices u_0, s_0, r_0.

$$(5.49)\quad D_{u_0} D_{[s_0} D_{r_0]} w_{q_x} = -\tfrac{1}{2} w_{p_x} D_{u_0} \overset{x}{K}{}^{\cdot\cdot\cdot\cdot p_x}_{s_0 r_0 q_x} - \tfrac{1}{2} \overset{x}{K}{}^{\cdot\cdot\cdot\cdot p_x}_{s_0 r_0 q_x} D_{u_0} w_{p_x};$$

$$(5.50)\quad D_{[u_0} D_{s_0} D_{r_0]} w_{q_x} = -\tfrac{1}{2} w_{p_x} D_{[u_0} \overset{x}{K}{}^{\cdot\cdot\cdot\cdot p_x}_{s_0 r_0] q_x} - \tfrac{1}{2} \overset{x}{K}{}^{\cdot\cdot\cdot\cdot p_x}_{[s_0 r_0 | q_x |}\, D_{u_0]} w_{p_x}.$$

Comparing (5.46) and (5.50), we see that

$$(5.51)\quad w_{p_x} D_{[u_0} \overset{x}{K}{}^{\cdot\cdot\cdot\cdot p_x}_{s_0 r_0] q_x} = 0.$$

Since w_{p_x} is an arbitrary vector of the x^{th} normal space, (5.51) tells us that (5.43) is true, and our lemma is proved.

1. *The Reason for the Imposition of the Condition* $m_{x+1} = 1$ *in Theorem V.* When we considered (5.29) we said that we added the assumption $m_{x+1} = 1$ to the statement of Theorem V because we could not conclude from (5.29) that (5.36) held if $m_{x+1} = 2$.[48] To justify this statement, we shall show now that there exists a set of vectors $\underset{B_1}{v}{}^{s_0}_{A_1}\, \underset{B_1}{u}{}^{r_0}_{A_1}\, \underset{B_1}{z}{}^{p_x}_{A_1}\, \theta_{s_0 r_0 r_{x+1} p_x}$ which satisfies (5.29) when $m_{x+1} = 2$, and which is not composed of null vectors.

For the case $m_{x+1} = 2$, $\beta_{x+1} = 2m_{x+1} + 1 = 5$, we can write equations

[47] I, pg. 113.

[48] See footnote 43.

(5.31) as

(5.52)

1. $H^{r_{s+1}}_{1\ 2}\theta_{r_{s+1}} + H^{r_{s+1}}_{2\ 1}\theta_{r_{s+1}} = 0$
2. $H^{r_{s+1}}_{1\ 3}\theta_{r_{s+1}} + H^{r_{s+1}}_{3\ 1}\theta_{r_{s+1}} = 0$
3. $H^{r_{s+1}}_{2\ 3}\theta_{r_{s+1}} + H^{r_{s+1}}_{3\ 2}\theta_{r_{s+1}} = 0$
4. $H^{r_{s+1}}_{1\ 4}\theta_{r_{s+1}} + H^{r_{s+1}}_{4\ 1}\theta_{r_{s+1}} = 0$
5. $H^{r_{s+1}}_{2\ 4}\theta_{r_{s+1}} + H^{r_{s+1}}_{4\ 2}\theta_{r_{s+1}} = 0$
6. $H^{r_{s+1}}_{1\ 5}\theta_{r_{s+1}} + H^{r_{s+1}}_{5\ 1}\theta_{r_{s+1}} = 0$
7. $H^{r_{s+1}}_{2\ 5}\theta_{r_{s+1}} + H^{r_{s+1}}_{5\ 2}\theta_{r_{s+1}} = 0$
8. $H^{r_{s+1}}_{3\ 4}\theta_{r_{s+1}} + H^{r_{s+1}}_{4\ 3}\theta_{r_{s+1}} = 0$
9. $H^{r_{s+1}}_{3\ 5}\theta_{r_{s+1}} + H^{r_{s+1}}_{5\ 3}\theta_{r_{s+1}} = 0$
10. $H^{r_{s+1}}_{4\ 5}\theta_{r_{s+1}} + H^{r_{s+1}}_{5\ 4}\theta_{r_{s+1}} = 0.$

Because none of the vectors $H^{r_{s+1}}_{B}$ are null vectors, and because there are two independent vectors among them, we can assume that

$$H^{r_{s+1}}_{B} = C_{B\ 1}\, H^{r_{s+1}}_{1} + D_{B\ 2}\, H^{r_{s+1}}_{2},\quad C_{1} = 1,\ C_{2} = 0,\ D_{1} = 0,\ D_{1} = 1. \tag{5.53}$$

Now let us consider the following system of equations, where our unknowns are $\psi_{1}, \psi_{1}, \psi_{2}, \psi_{2}$:

$$\begin{aligned} &1.\quad H^{r_{s+1}}_{2}\psi_{r_{s+1}} + H^{r_{s+1}}_{1}\psi_{r_{s+1}} = 0,\\ &2.\quad H^{r_{s+1}}_{1}\psi_{r_{s+1}} = 0,\\ &3.\quad H^{r_{s+1}}_{2}\psi_{r_{s+1}} = 0. \end{aligned} \tag{5.54}$$

There surely exists a non-null solution of (5.54) because there are fewer equations than unknowns.

We specify our non-null solution of (5.52) as follows: Let $\psi_{r_{s+1}}$, $\psi_{r_{s+1}}$ be a non-null solution of (5.54). Let $\psi_{r_{s+1}} = C\psi_{r_{s+1}} + D\psi_{r_{s+1}}$, $\psi_{r_{s+1}} = C\psi_{r_{s+1}} + D\psi_{r_{s+1}}$, and $\psi_{r_{s+1}} = C\psi_{r_{s+1}} + D\psi_{r_{s+1}}$. To check the statement that this set of vectors $\psi_{r_{s+1}}$ is a solution of (5.52), we note that (5.52)-1 is satisfied because it is the same equation as (5.54)-1; (5.52)-2, $\cdots$, (5.52)-10 are satisfied because they can be made to depend on (5.54). For instance, we have for (5.52)-2

$$\begin{aligned} &H^{r_{s+1}}_{1\ 3}\psi_{r_{s+1}} + H^{r_{s+1}}_{3\ 1}\psi_{r_{s+1}}\\ &\quad = 2CH^{r_{s+1}}_{3\ 1}\psi_{r_{s+1}} + D(H^{r_{s+1}}_{3\ 1}\psi_{r_{s+1}} + H^{r_{s+1}}_{2\ 1}\psi_{r_{s+1}}) = 0, \end{aligned} \tag{5.55}$$

and (5.52)-8 becomes

$$
\begin{aligned}
&H^{r_{s+1}}_{3}\psi_{r_{s+1}} + H^{r_{s+1}}_{4}\psi_{r_{s+1}} = 2CCH^{r_{s+1}}_{3\,4\,1}\psi_{r_{s+1}} \\
&+ (CD + CD)(H^{r_{s+1}}_{1}\psi_{r_{s+1}} + H^{r_{s+1}}_{2}\psi_{r_{s+1}}) + 2DDH^{r_{s+1}}_{3\,4\,2}\psi_{r_{s+1}} = 0.
\end{aligned} \tag{5.56}
$$

An additional assumption of the type "Condition A for H shall be satisfied" would put a set of additional conditions on the constants C, D of (5.53), but this same illustration of a non-null solution of (5.52) would still hold.

Of course one might be able to conclude that the vectors $\theta_{r_{s+1}} = v^{r_0}\, u^{r_0}\, z^{p_s}\, \theta_{s_0 r_0 r_{s+1} p_s}$ are null vectors if one had relationships for them in addition to those of (5.29). However, it seems to be impossible to get such additional relationships by our methods of transvection, and so we have no choice but to impose the condition $m_{s+1} = 1$ in Theorem V.

VI. The Significance of the Conditions Imposed upon the Affinors $H^{s}_{r_0 p_{s-1}}$

A. The Algebraic Expression of the Conditions. We shall find it convenient to express the affinors $H^{s}_{r_0 p_{s-1}}$ in terms of vectors and affinors of valence two.

If we use the Frenet equations (1.9) and also the fact that H^{s}_{λ} lies with the index λ in the $(s-1)^{st}$ normal space,[49] we can say that

$$
i^{p_{s-1}}_{q_{s-1}}\, i^{t_s}_{s}\, H^{s}_{r_0 p_{s-1}} = i^{p_{s-1}}_{q_{s-1}}\, i^{t_s}_{s}(D_{r_0} B^{s-1}_{p_{s-1}} + H^{s-1}_{r_0 p_{s-1}}) = i^{p_{s-1}}_{q_{s-1}}\, i^{t_s}_{s} D_{r_0} B^{s-1}_{p_{s-1}}. \tag{6.1}
$$

Here the vectors $i^{p_{s-1}}_{q_{s-1}}$ and $i^{t_s}_{s}$ are the mutually perpendicular unit vectors of the $(s-1)^{st}$ and s^{th} normal spaces discussed in Section I A. Now we transvect in (6.1) by $i^{q_{s-1}}_{r_{s-1}}$, $i^{\lambda}_{t_s}$ and we use (1.6), (3.6), (3.7).

$$
\begin{aligned}
&i^{p_{s-1}}_{q_{s-1}}\, i^{q_{s-1}}_{r_{s-1}}\, i^{t_s}_{s}\, i^{\lambda}_{t_s}\, H^{s}_{r_0 p_{s-1}} \\
&= B^{p_{s-1}}_{r_{s-1}} B^{\lambda}_{s} H^{s}_{r_0 p_{s-1}} = H^{s}_{r_0 r_{s-1}}{}^{\lambda} = i^{p_{s-1}}_{q_{s-1}}\, i^{q_{s-1}}_{r_{s-1}}\, i^{t_s}_{s}\, i^{\lambda}_{t_s} D_{r_0} B^{s-1}_{p_{s-1}}.
\end{aligned} \tag{6.2}
$$

[49] See (3.6).

If we modify the right hand member of (6.2) by using (1.6) again, we have

$$(6.3)\qquad \overset{x}{H}_{r_0 r_{x-1}}{}^{\cdot\cdots\cdot}{}^{\lambda} = \underset{q_{x-1}}{i^{p_{x-1}}}\, \underset{t_x}{i^{q_{x-1}}_{r_{x-1}}}\, \underset{t_x}{i^{\lambda}_{x}} D_{r_0}\Big(\underset{t_{x-1}}{i^{p_{x-1}}_{p_{x-1}}}\, \underset{t_{x-1}}{i^{\pi}}\Big) = \underset{t_x}{i^{t_{x-1}}_{r_{x-1}}}\, \underset{t_x}{i^{t_x}_{x}}\, \underset{t_x}{i^{\lambda}} D_{r_0}\, \underset{t_{x-1}}{i^{\pi}}\,,$$

or

$$(6.4)\qquad \overset{x}{H}_{r_0 s_{x-1}}{}^{\cdot\cdots\cdot}{}^{\lambda} = -\, \underset{t_{x-1}}{i^{t_{x-1}}_{s_{x-1}}}\, \underset{t_{x-1}}{i^{\pi}}\, (D_{r_0} \underset{t_x}{i^{t_x}_{\kappa}})\, \underset{t_x}{i^{\lambda}} = -\, \overset{x-1}{B}{}^{\pi}_{s_{x-1}}\, \overset{0}{B}{}^{\mu}_{r_0} (D_{\mu} \underset{t_x}{i^{t_x}_{\kappa}})\, \underset{t_x}{i^{\lambda}}\,.$$

If we define

$$(6.5)\qquad \underset{t_x}{h_{r_0 s_{x-1}}} = -\, \overset{0}{B}{}^{\mu}_{r_0}\, \overset{x-1}{B}{}^{\kappa}_{s_{x-1}}\, D_{\mu} \underset{t_x}{i^{t_x}_{\kappa}}\,,$$

(6.4) becomes

$$(6.6)\qquad \overset{x}{H}_{r_0 s_{x-1}}{}^{\cdot\cdots\cdot}{}^{\lambda} = \underset{t_x}{h_{r_0 s_{x-1}}}\, \underset{t_x}{i^{\lambda}}\,.$$

We note that this equation is familiar to us for the case $x = 1$.[50] If, in addition, we are considering a $V_{n-1} \subset V_n$, so that $m = 1$, then we have only one tensor $\underset{t_1}{h_{r_0 s_0}}$, and we usually call it the second fundamental tensor for the V_{n-1}.

We shall be concerned with the rank of the affinor $\overset{x}{H}_{r_0 p_{x-1}}{}^{\cdot\cdots\cdot}{}^{q_x}$, and in that connection, if we consider the definition of condition 0 for $\overset{x}{H}$,[51] we see that this definition makes at least two requirements on H. First, the equations (2.1) tell us that all the vectors that can be formed from $\overset{x}{H}_{r_0 p_{x-1}}{}^{\cdot\cdots\cdot}{}^{q_x}$ by transvection on the indices p_{x-1}, q_x lie in the $\underset{x}{\beta}$-dimensional space which is part of the 0^{th} normal space but which is normal to the $\underset{x}{\alpha}$ independent vectors $\underset{A}{\overset{x}{u}}{}^{r_0}$. Furthermore, all these vectors which are arrived at by transvection on p_{x-1}, q_x cannot be included in a $(\underset{x}{\beta} - 1)$-dimensional space, because if they could, there would be $\underset{x}{\alpha} + 1$ vectors of the type $\underset{A}{\overset{x}{u}}{}^{r_0}$ rather than $\underset{x}{\alpha}$. But by the definition of the rank of an affinor with respect to a specified index,[52] this means that

[50] II, pg. 82.
[51] II, pg. 61.
[52] See Section 11.
[53] I, pg. 19.

LEMMA VI. 1. If condition 0 for $\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}$ is satisfied, then $\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}$ is of rank β with respect to the index r_0.

In exactly the same way, (2.3) tells us that

LEMMA VI.2. If condition 0 for $\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}$ is satisfied, then $\overset{x}{H}_{r_0 p_{x-1}}^{\cdot\cdot\cdot q_x}$ is of rank β with respect to the index p_{x-1}.

1. *The Case where* $m = 1$. When $\underset{x}{m} = 1$, there is only one unit vector in the x^{th} normal space. If we call it $\underset{x}{i^{\lambda}}$, we can write (6.6) as

$$\overset{x}{H}_{r_0 s_{x-1}}^{\cdot\cdot\cdot\lambda} = h_{r_0 s_{x-1}} \underset{x}{i^{\lambda}}. \tag{6.7}$$

If condition A for $\overset{x}{H}$ is satisfied, Lemma II.2 tells us that $\underset{x}{\beta} \geq A + 1$, and since in that case condition 0 for $\overset{x}{H}$ is also satisfied, Lemmas VI.1 and VI.2 tell us that the rank of $\overset{x}{H}_{r_0 s_{x-1}}^{\cdot\cdot\cdot p_x}$ with respect to either index r_0 or index p_{x-1} is $\geq A + 1$. Using (6.7), we can state this result as

LEMMA VI.3. If condition A for $\overset{x}{H}$ is satisfied, then the rank of $h_{r_0 p_{x-1}}$ with respect to either index is $\geq A + 1$.

Conversely, suppose that the rank of $h_{r_0 s_{x-1}}$ is R with respect to both indices, $R \geq A + 1$. When considered as a matrix, $h_{r_0 s_{x-1}}$ is equivalent under non-singular linear transformations to any other $\underset{0}{m} \times \underset{x-1}{m}$ matrix of the same rank R.[54] We can consider $h_{r_0 s_{x-1}}$ as equivalent, for instance, to an $\underset{0}{m} \times \underset{x-1}{m}$ diagonal matrix:

$$\| h_{r_0 s_{x-1}} \| \overset{*}{=} \left\| \begin{matrix} h_{1'1'} & 0 & 0 & \cdot & 0 & 0 & \cdot\cdot & 0 \\ 0 & h_{2'2'} & 0 & \cdot & 0 & 0 & \cdot\cdot & 0 \\ 0 & 0 & h_{3'3'} & \cdot & 0 & 0 & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ 0 & 0 & 0 & \cdot & h_{R'R'} & 0 & \cdot\cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & 0 & \cdot\cdot & 0 \end{matrix} \right\| \begin{matrix} \uparrow \\ \\ \\ \\ \\ \\ \\ \downarrow \end{matrix} \underset{x-1}{m}, \tag{6.8}$$

$$\leftarrow \underset{0}{m} \rightarrow$$

$$h_{1'1'} h_{2'2'} \cdots h_{R'R'} \neq 0.$$

We can select quite easily $\underset{x}{\alpha} = \underset{0}{m} - R$ vectors $\overset{x}{u}_{A}^{r_0}$, $A = 1, \cdots, \underset{x}{\alpha}$,

[54] L. E. Dickson, "Modern Algebraic Theories," Sanborn, 1930, pg. 57.

such that

$$\underset{A}{\overset{x}{u}}{}^{r_0'} h_{r_0' s_{x-1}'} = 0. \tag{6.9}$$

For instance, we might choose $\left\| \underset{A}{\overset{x}{u}}{}^{r_0'} \right\| \overset{*}{=} \left\| \underbrace{0, 0, \ldots, 0}_{0}, \underbrace{1, 0, \ldots, 0}_{A} \right\|$. In the same way we can choose $\gamma = m - R$ vectors $\underset{C}{\overset{x}{y}}{}^{s_{x-1}'}$, $C = 1, \cdots, \gamma$, such that

$$\underset{C}{\overset{x}{y}}{}^{s_{x-1}'} h_{r_0' s_{x-1}'} = 0. \tag{6.10}$$

We might take $\left\| \underset{C}{\overset{x}{y}}{}^{s_{x-1}'} \right\| \overset{*}{=} \left\| \underbrace{0, \ldots, 0}_{m-x-1}, \underbrace{1, 0, \ldots, 0}_{\gamma} \right\|$. If we choose our $\underset{B}{\overset{x}{v}}{}^{r_0'}$, $B = 1, \ldots, R = \beta$, as $\left\| \underset{B}{\overset{x}{v}}{}^{r_0'} \right\| \overset{*}{=} \left\| \underbrace{0, \ldots, 0}_{0}, \underbrace{1, 0, \ldots, 0}_{B} \right\|$, and our $\underset{B}{\overset{x}{z}}{}^{s_{x-1}'}$, $B = 1, \ldots, R$, as $\left\| \underset{B}{\overset{x}{z}}{}^{s_{x-1}'} \right\| \overset{*}{=} \left\| \underbrace{0, \ldots, 0}_{m-x-1}, \underbrace{1, 0, \ldots, 0}_{B} \right\|$, then

$$\underset{B_1 B_2}{\overset{x}{v}}{}^{r_0'} \overset{x}{z}{}^{s_{x-1}'} h_{r_0' s_{x-1}'} \begin{cases} = 0, & B_1 \neq B_2 \\ \neq 0, & B_1 = B_2. \end{cases} \tag{6.11}$$

Now if we transform from the coordinate system (r_0') to (r_0) and from (s_{x-1}') to (s_{x-1}), then we shall have vectors $\underset{A}{\overset{x}{u}}{}^{r_0}$, $\underset{B}{\overset{x}{v}}{}^{r_0}$, $\underset{B}{\overset{x}{z}}{}^{s_{x-1}}$, $\underset{C}{\overset{x}{y}}{}^{s_{x-1}}$ satisfying (2.1), (2.2), and (2.3), and we can say that condition 0 for H is satisfied. Because $\beta = R \geq A + 1$, Lemma II.2 permits us to say in addition that condition A for H is satisfied. We summarize our work in

LEMMA VI.4. If $h_{r_0 s_{x-1}}$ is of rank $\beta \geq A + 1$ with respect to both indices, then condition A for H is satisfied.

Lemmas VI.3 and VI.4 together enable us to say that

THEOREM VI A 1. If $m = 1$, then the statements "Condition A for H is satisfied" and "$h_{r_0 s_{x-1}}$ is of rank $\beta \geq A + 1$ with respect to both indices" are equivalent statements.

a. A Theorem of T. Y. Thomas. A theorem of T. Y. Thomas states that if the second fundamental form of a V_{n-1} imbedded in an R_n is of rank ≥ 4, then the Codazzi equation is a consequence of the Gauss equation.[55]

[55] T. Y. Thomas, Acta Mathematica, 67, 1935, pg. 189.

Our Theorem V tells us that if condition 3 for $\overset{x}{H}$ is satisfied for a $V_{n-1} \subset V_n$, then the Codazzi equation $\overset{1}{N}$ is a consequence of the Gauss equation $\overset{0}{N}$. Theorem VI A 1 tells us however that Thomas' condition is exactly the same as ours, and thus Thomas' theorem is a special case of our Theorem V.

2. *The Case where* $m = 2$. When $m = 2$, one can select two independent unit vectors in the x^{th} normal space. We do this, and if we call these vectors $i^{r_x}_{1_x}$, $i^{r_x}_{2_x}$ we can write (6.6) as

$$\overset{x}{H}{}^{\cdot}_{r_0 p_{x-1}}{}^{\cdot}{}^{r_x} = h_{r_0 p_{x-1}} i^{r_x}_{1_x} + h_{r_0 p_{x-1}} i^{r_x}_{2_x}. \tag{6.12}$$

We shall assume that condition A for H is satisfied but that condition $A + 1$ is not, and we shall see what this assumption tells us about the affinors $\overset{1_x}{h}$ and $\overset{2_x}{h}$.

There exist $\underset{x}{\alpha}$ independent vectors $\overset{x}{u}{}^{r_0}_{A}$, $A = 1, \cdots, \underset{x}{\alpha}$; $\underset{x}{\gamma}$ independent vectors $\overset{x}{y}{}^{p_{x-1}}_{C}$, $C = 1, \cdots, \underset{x}{\gamma}$; $\underset{x}{\beta} = \underset{0}{m} - \underset{x}{\alpha}$ independent vectors $\overset{x}{v}{}^{r_0}_{B}$, $B = 1, \cdots, \underset{x}{\beta}$; and $\underset{x}{\beta} = \underset{x-1}{m} - \underset{x}{\gamma}$ independent vectors $\overset{x}{z}{}^{p_{x-1}}_{B}$ such that

$$\overset{x}{u}{}^{r_0}_{A} \overset{t_x}{h}_{r_0 p_{x-1}} = 0, \qquad t_x = 1_x, 2_x; \tag{6.13}$$

$$\overset{x}{y}{}^{p_{x-1}}_{C} \overset{t_x}{h}_{r_0 p_{x-1}} = 0, \qquad t_x = 1_x, 2_x; \tag{6.14}$$

$$\overset{x}{v}{}^{r_0}_{B_1} \overset{x}{z}{}^{p_{x-1}}_{B_2} \overset{t_x}{h}_{r_0 p_{x-1}} = 0, \qquad B_1 \neq B_2, \qquad t_x = 1_x, 2_x; \tag{6.15}$$

$$\overset{x}{v}{}^{r_0}_{B} \overset{x}{z}{}^{p_{x-1}}_{B} \overset{1_x}{h}_{r_0 p_{x-1}}, \quad \overset{x}{v}{}^{r_0}_{B} \overset{x}{z}{}^{p_{x-1}}_{B} \overset{2_x}{h}_{r_0 p_{x-1}} \text{ not both zero}, \quad B = 1, \cdots, \underset{x}{\beta}. \tag{6.16}$$

We shall designate by (r_0') a coordinate system of the V_m whose measuring vectors are the unit vectors in the directions of the $\underset{0}{m}$ independent vectors $\overset{x}{v}{}^{r_0}_{B}$, $\overset{x}{u}{}^{r_0}_{A}$. In particular, we designate the measuring vectors by $i^{r_0'}_{s_0'}$, p_0', r_0', $s_0' = 1', 2', \cdots, m'$, and we choose $i^{r_0'}_{B'} \parallel \overset{x}{v}{}^{r_0'}_{B}$, $B' = 1', \cdots, \underset{x}{\beta}'$; $B = 1, \cdots, \underset{x}{\beta}$; and $i^{r_0'}_{(\beta+A)'} \parallel \overset{x}{u}{}^{r_0'}_{A}$, $(\underset{x}{\beta} + A)' = (\underset{x}{\beta} + 1)'$,

$(\beta + 2)', \cdots, m', A = 1, \cdots, \alpha$. This coordinate system is not necessarily holonomic.

In a similar way we select a coordinate system in the $(x-1)^{st}$ normal space whose measuring vectors are unit vectors in the directions of the m independent vectors $z^{p_{x-1}}, y^{p_{x-1}}$, and designate this coordinate system by $(p_{x-1}`)$. In particular, we designate the measuring vectors this time by $j^{p_{x-1}`}, p_{x-1}`, q_{x-1}`, r_{x-1}` = 1`, 2`, \cdots, m`$, and we choose $j^{p_{x-1}`} \| z^{p_{x-1}`}$, $B` = 1`, \cdots, \beta`; B = 1, \cdots, \beta; j^{p_{x-1}`} \| y^{p_{x-1}`}$, $(\beta + C)` = (\beta + 1)`$, $(\beta + 2)`, \cdots, m`, C = 1, \cdots, \gamma$. This coordinate system also need not be holonomic.

When expressed in the coordinate systems (r_0') and $p_{x-1}`)$, the matrices representing the affinors h and h take on a simple form. If we agree that for the affinors $h_{r_0 p_{x-1}}$ the first index shall represent columns while the second shall represent rows, the equations (6.13) tell us that the last α columns of the matrices are composed of zeros, and the equations (6.14) tell us that the last γ rows are likewise composed of zeros. Of the terms in the remaining β columns and β rows, we learn from (6.15) that only the diagonal terms do not vanish. All this information enables us to write the matrices for these affinors as

$$(6.17) \qquad \left\| h_{r_0' p_{x-1}`} \right\| \doteq \begin{array}{|cccccc|} \hline h_{1'1`} & 0 & \cdot & 0 & 0 & 0 \\ 0 & h_{2'2`} & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & h_{\beta'\beta`} & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & 0 & 0 \\ \hline \end{array} \begin{array}{l} \uparrow \\ \\ \\ m \\ \\ \\ \downarrow \end{array}$$

$$\leftarrow \qquad m \qquad \rightarrow$$

Because of (6.16) we know too that $h_{B'B`}$ and $h_{B'B`}$ do not both vanish.

Thus far we have only used that part of our assumption that tells us

that condition 0 for $\overset{x}{H}$ is satisfied. Because condition A for $\overset{x}{H}$ is satisfied we know that there are at most $\beta - A - 1$ vectors in the set $\underset{B}{v^{r_0}}\underset{B}{z^{p_x-1}}\overset{x}{H}_{r_0 p_x - 1}^{r_x}$, $B = 1, \cdots, \beta$, which are parallel to each other. If there were more than $\beta - A - 1$ vectors of that set parallel to each other, it would be possible to find a subset of $\beta - A$ vectors in the set $\underset{B}{v^{r_0}}\underset{B}{z^{p_x-1}}\overset{x}{H}_{r_0 p_x - 1}^{r_x}$ which would include but one independent vector, and our assumption that condition A for $\overset{x}{H}$ is satisfied would be violated. We have assumed also that condition $A + 1$ for $\overset{x}{H}$ is not satisfied, and this assumption tells us that there are at least $\beta - A - 1$ vectors in the set $\underset{B}{v^{r_0}}\underset{B}{z^{p_x-1}}\overset{x}{H}_{r_0 p_x - 1}^{r_x}$ which are parallel to each other. If this were not true, there would always be two independent vectors in every subset of $\beta - A - 1$ vectors of the set $\underset{B}{v^{r_0}}\underset{B}{z^{p_x-1}}\overset{x}{H}_{r_0 p_x - 1}^{r_x}$ that we considered, and we would contradict the assumption that condition $A + 1$ for $\overset{x}{H}$ is not satisfied. Therefore, our two assumptions together tell us that there exists a subset of $\beta - A - 1$ vectors which are parallel to each other in the set of β vectors $\underset{B}{v^{r_0}}\underset{B}{z^{p_x-1}}\overset{x}{H}_{r_0 p_x - 1}^{r_x}$, and that there does not exist any other subset of more vectors whose members are also parallel to each other.

To make use of the result that there exists this subset of $\beta - A - 1$ parallel vectors, we write a necessary and sufficient condition that two vectors, $\underset{B_1}{v^{r_0}}\underset{B_1}{z^{p_x-1}}\overset{x}{H}_{r_0 p_x - 1}^{r_x}$ and $\underset{B_2}{v^{r_0}}\underset{B_2}{z^{p_x-1}}\overset{x}{H}_{r_0 p_x - 1}^{r_x}$, shall be parallel or linearly dependent. We use (6.12), and remember that the vectors $i^{r_x}_{1_x}$, $i^{r_x}_{2_x}$ written there are independent vectors. We can say then that the vectors $\underset{B_1}{v^{r_0}}\underset{B_1}{z^{p_x-1}}\overset{x}{H}_{r_0 p_x - 1}^{r_x}$ and $\underset{B_2}{v^{r_0}}\underset{B_2}{z^{p_x-1}}\overset{x}{H}_{r_0 p_x - 1}^{r_x}$ are parallel if and only if the determinant (6.18) vanishes.

$$(6.18)\qquad \begin{vmatrix} \underset{B_1}{v^{r_0}}\underset{B_1}{z^{p_x-1}}\overset{1_x}{h}_{r_0 p_x - 1} & \underset{B_1}{v^{r_0}}\underset{B_1}{z^{p_x-1}}\overset{2_x}{h}_{r_0 p_x - 1} \\ \underset{B_2}{v^{r_0}}\underset{B_2}{z^{p_x-1}}\overset{1_x}{h}_{r_0 p_x - 1} & \underset{B_2}{v^{r_0}}\underset{B_2}{z^{p_x-1}}\overset{2_x}{h}_{r_0 p_x - 1} \end{vmatrix} = \underset{B_1}{v^{B_1}}\underset{B_1}{z^{B_1}}\underset{B_2}{v^{B_2}}\underset{B_2}{z^{B_2}} \begin{vmatrix} \overset{1_x}{h}_{B_1 B_1} & \overset{2_x}{h}_{B_1 B_1} \\ \overset{1_x}{h}_{B_2 B_2} & \overset{2_x}{h}_{B_2 B_2} \end{vmatrix},$$

Where in VI A 1 we were led to a consideration of single matrices, we can apply here some of the results of the theory of classical algebra which deal with pencils of matrices.[56] Equations (6.17) show us that the affinors $\overset{1_x}{h}_{r_0 p_{x-1}}$ and $\overset{2_x}{h}_{r_0 p_{x-1}}$ can be considered as representing a pair of bilinear forms in $\beta + \beta$ variables. Other matrices which can represent this pair of bilinear forms in $\beta + \beta$ variables are arrived at by striking out the last α columns and γ rows of the matrices (6.17). If we designate these new matrices by $\overset{1_x}{h}_{r_0 p_{x-1}}$, $\overset{2_x}{h}_{r_0 p_{x-1}}$, $r_0 = 1', \cdots, \beta'$, $p_{x-1} = 1, \cdots, \beta$; we can write them as

$$(6.19)\qquad \| \overset{t_x}{h}_{r_0 p_{x-1}} \| \overset{*}{=} \begin{Vmatrix} \overset{t_x}{h}_{1,1} & 0 & \cdots & 0 \\ 0 & \overset{t_x}{h}_{2,2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \overset{t_x}{h}_{\beta,\beta} \end{Vmatrix}, \quad t_x = 1_x, 2_x .$$

When we consider the matrix $\| \mu \overset{1_x}{h}_{r_0 p_{x-1}} + \lambda \overset{2_x}{h}_{r_0 p_{x-1}} \|$, we can say that it is a non-singular matrix, that is that its determinant does not vanish identically in λ and μ, because (6.16) tells us that $\overset{1_x}{h}_{\beta,\beta}$ and $\overset{2_x}{h}_{\beta,\beta}$ do not both vanish.

The elementary divisors of this non-singular matrix,

$$\| \mu \overset{1_x}{h}_{r_0 p_{x-1}} + \lambda \overset{2_x}{h}_{r_0 p_{x-1}} \|$$

$$(6.20)\qquad \overset{*}{=} \begin{Vmatrix} \mu \overset{1_x}{h}_{1,1} + \lambda \overset{2_x}{h}_{1,1} & 0 & \cdots & 0 \\ 0 & \mu \overset{1_x}{h}_{2,2} + \lambda \overset{2_x}{h}_{2,2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mu \overset{1_x}{h}_{\beta,\beta} + \lambda \overset{2_x}{h}_{\beta,\beta} \end{Vmatrix}$$

are all linear because we have only diagonal terms, and the fact that

[56] This theory is discussed in many standard algebra texts. In particular, one might mention H. W. Turnbull and A. Aitken, "An Introduction to the Theory of Canonical Matrices," Blackie and Son, London, 1933, Chapter IX; M. Bocher, "Introduction to Higher Algebra," MacMillan, New York, 1936; and L. E. Dickson, "Modern Algebraic Theories," Sanborn, Chicago, 1930.

exactly $\beta_x - A - 1$ of the vectors $v^{r_{\beta}}_{\beta} g^{p_{x-1}}_{\beta} H_{r_{\beta p_{x-1}}}$ are parallel to each other tells us through (6.18) that one of the elementary divisors is repeated $\beta_x - A - 1$ times. Further, because there is no subset of more than $\beta_x - A - 1$ parallel vectors in the set $v^{r_{\beta}}_{\beta} g^{p_{x-1}}_{\beta} H_{r_{\beta p_{x-1}}}$, no elementary divisor is repeated more than $\beta_x - A - 1$ times.

We shall state the conclusions we have drawn from our original assumptions in the following lemma:

LEMMA VI.5. If $m_x = 2$, and if condition A for H_x is satisfied but condition $A + 1$ is not, then the pair of affinors h_{1x}, h_{2x} defined by (6.5) and (6.12) can be considered as representing a non-singular pair of bilinear forms in $\beta_x + \beta_x$ variables. The elementary divisors for this pair of bilinear forms are all linear, and the most frequently repeated elementary divisor is repeated exactly $\beta_x - A - 1$ times.

If we choose to, we can also state Lemma VI.5 in the following words:

LEMMA VI. 5—Alternate Form—If $m_x = 2$, and if condition A for H_x is satisfied but condition $A + 1$ is not, then the pencil of matrices $\mu h_{1x} + \lambda h_{2x}$, h_{1x} and h_{2x} being defined by (6.5) and (6.12), has a set of α_x minimal numbers all zero, a second set of γ_x minimal numbers all zero, and a non-singular square core of β_x rows and columns whose elementary divisors are all linear, the most frequently repeated elementary divisor being repeated $\beta_x - A - 1$ times.

Conversely, if $m_x = 2$, and if we know that the affinors h_{1x} and h_{2x} have the properties stated in the conclusion of Lemma VI.5, does it follow that condition A for H_x is satisfied but that condition $A + 1$ is not?

The fact that the affinors h_{1x} and h_{2x} can be considered as representing a pair of bilinear forms in $\beta_x + \beta_x$ variables means that there exist pairs of coordinate systems such that the matrices h_{1x} and h_{2x} have their last α_x columns and γ_x rows all zeros when expressed in these coordinate systems.

Let (r_0') and (p_{x-1}') be such a pair of coordinate systems. Then

$$(6.21)\qquad \| \overset{t_x}{h}_{r_0' p_{x-1}'} \| \triangleq \left\| \begin{array}{cccccccc} \overset{t_x}{h}_{1'1'} & \cdots & \overset{t_x}{h}_{\beta_x' 1'} & 0 & \cdot & \cdot & 0 & \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \overset{t_x}{h}_{1'\beta_x'} & \cdots & \overset{t_x}{h}_{\beta_x'\beta_x'} & 0 & \cdot & \cdot & 0 & \\ 0 & \cdots & 0 & 0 & \cdot & \cdot & 0 & \uparrow \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma_x \\ 0 & \cdots & 0 & 0 & \cdot & \cdot & 0 & \downarrow \\ & & & \longleftarrow & \alpha_x & \longrightarrow & & \end{array} \right\| \qquad t_x = 1_x, 2_x.$$

Now we can select α_x independent vectors $\overset{x}{u}{}^{r_0'}_{A}$, $A = 1, \cdots, \alpha_x$ such that

$$(6.22)\qquad \overset{x}{u}{}^{r_0'}_{A} \overset{t_x}{h}_{r_0' p_{x-1}'} = 0.$$

Indeed, the vectors $\overset{x}{u}{}^{r_0'}_{A}$ may be chosen by the following scheme: $\left\| \overset{x}{u}{}^{r_0'}_{A} \right\| \triangleq \left\| \underbrace{0, \cdots, 0}_{m}, \overset{A}{\overbrace{1, 0, \cdots, 0}} \right\|$. Similarly, we choose γ_x independent vectors of the $(x-1)^{st}$ normal space by the following scheme: $\left\| \overset{x}{y}{}^{p_{x-1}'}_{C} \right\| \triangleq \left\| \underbrace{0, \cdots, 0}_{m}, \overset{C}{\overbrace{1, 0, \cdots, 0}} \right\|$ and notice that

$$(6.23)\qquad \overset{x}{y}{}^{p_{x-1}'}_{C} \overset{t_x}{h}_{r_0' p_{x-1}'} = 0.$$

We know in addition that the elementary divisors for the non-singular pair of bilinear forms in $\beta_x + \beta_x$ variables represented by $\overset{1_x}{h}$ and $\overset{2_x}{h}$ are all linear and that the most frequently repeated elementary divisor is repeated exactly $\beta_x - A - 1$ times. But because non-singular pairs of bilinear forms are equivalent if they have the same elementary divisors, this means that we can find non-singular transformations which leave unchanged the last $\alpha_x = m_0 - \beta_x$ coordinates of the coordinate system (r_0') and the last $\gamma_x = m_{x-1} - \beta_x$ coordinates of the coordinate

system (p'_{x-1}), and which lead to coordinate systems (r_0) and (p_{x-1}) in which $\overset{1_x}{h}$ and $\overset{2_x}{h}$ are such that

$$\left\| \overset{1_x}{\mu h}_{r_0 p_{x-1}} + \overset{2_x}{\lambda h}_{r_0 p_{x-1}} \right\|$$

$$\overset{*}{=} \left\| \begin{matrix} \overset{1_x}{\mu h}_{11} + \overset{2_x}{\lambda h}_{11} & 0 & \cdot & 0 & 0 & \cdot & 0 \\ 0 & \overset{1_x}{\mu h}_{22} + \overset{2_x}{\lambda h}_{22} & \cdot & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \overset{1_x}{\mu h}_{\beta_x \beta_x} + \overset{2_x}{\lambda h}_{\beta_x \beta_x} & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & 0 & \cdot & 0 \end{matrix} \right\| . \tag{6.24}$$

Here none of the first β_x diagonal terms are identically vanishing, and a set of $\beta_x - A - 1$ of them are multiples of each other. No more numerous set of these terms has the property that each term is a multiple of every other term.

We can select β_x vectors $\overset{x}{v}{}^{r_0}_{B}$, $B = 1, \cdots, \beta_x$, which are independent of each other and independent of the vectors $\overset{x}{u}{}^{r_0}_{A}$, and also β_x vectors $\overset{x}{z}{}^{p_{x-1}}_{B}$ of the $(x-1)^{st}$ normal space which are independent of each other and independent of the vectors $\overset{x}{y}{}^{p_{x-1}}_{C}$. Further, these vectors can be chosen so that they have the properties

$$\overset{x}{v}{}^{r_0}_{B_1} \overset{x}{z}{}^{p_{x-1}}_{B_2} \overset{t_x}{h}_{r_0 p_{x-1}} = 0, \qquad B_1 \neq B_2, \qquad t_x = 1_x, 2_x; \tag{6.25}$$

$$\overset{x}{v}{}^{r_0}_{B} \overset{x}{z}{}^{p_{x-1}}_{B} \overset{1_x}{h}_{r_0 p_{x-1}}, \ \overset{x}{v}{}^{r_0}_{B} \overset{x}{z}{}^{p_{x-1}}_{B} \overset{2_x}{h}_{r_0 p_{x-1}} \quad \text{not both zero, } B = 1, \cdots, \beta_x. \tag{6.26}$$

A suitable scheme for choosing the vectors $\overset{x}{v}{}^{r_0}_{B}$ and $\overset{x}{z}{}^{p_{x-1}}_{B}$ is the following:

$$\left\| \overset{x}{v}{}^{r_0}_{B} \right\| \overset{*}{=} \left\| \underbrace{0, \cdots, 0, 1, 0, \cdots, 0}_{r_0} \right\|, \quad \left\| \overset{x}{z}{}^{p_{x-1}}_{B} \right\| \overset{*}{=} \left\| \underbrace{0, \cdots, 0, 1, 0, \cdots, 0}_{p_{x-1}} \right\|.$$

We consider the set of $\underset{x}{\beta}$ vectors $v^{r_0}_{B}\,z^{p_x-1}_{B}\,H^{x}_{r_0 p_x-1}{}^{r_x}$, and we note that the vanishing of the determinant

$$\begin{vmatrix} h^{1}_{B_1 B_1} & h^{2}_{B_1 B_1} \\ h^{1}_{B_2 B_2} & h^{2}_{B_2 B_2} \end{vmatrix} \tag{6.27}$$

is the necessary and sufficient condition that the vectors $v^{r_0}_{B_1}\,z^{p_x-1}_{B_1}\,H^{x}_{r_0 p_x-1}{}^{r_x}$ and $v^{r_0}_{B_2}\,z^{p_x-1}_{B_2}\,H^{x}_{r_0 p_x-1}{}^{r_x}$ shall be parallel.[57] Then because $\underset{x}{\beta} - A - 1$ of the diagonal terms of (6.24) are multiples of each other, we see that $\underset{x}{\beta} - A - 1$ of the vectors of the set $v^{r_0}_{B}\,z^{p_x-1}_{B}\,H^{x}_{r_0 p_x-1}{}^{r_x}$ are parallel to each other, and that there is no group of more than $\underset{x}{\beta} - A - 1$ parallel vectors in the set.

Finally, because (6.22) and (6.23) are tensor relationships and carry over for the unprimed coordinates we can conclude from (6.22), (6.23), (6.25), (6.26), (6.12), and from our consideration of the set of vectors $v^{r_0}_{B}\,z^{p_x-1}_{B}\,H^{x}_{r_0 p_x-1}{}^{r_x}$ that condition A for $\overset{x}{H}$ is satisfied but that condition $A + 1$ is not. We state our conclusion as

LEMMA VI.6. If $m = 2$, and if the affinors $\overset{1_x}{h}$ and $\overset{2_x}{h}$ defined by (6.5) and (6.12) represent a non-singular pair of bilinear forms in $\underset{x}{\beta} + \underset{x}{\beta}$ variables whose elementary divisors are all linear, the most frequently repeated elementary divisor being repeated $\underset{x}{\beta} - A - 1$ times, then condition A for $\overset{x}{H}$ is satisfied, but condition $A + 1$ is not.

If we choose to, we can state Lemma VI.6 in an alternate form.

LEMMA VI.6—Alternate Form—If $m = 2$, and if the pencil of matrices $\mu h^{1_x}_{r_0 p_x-1} + \lambda h^{2_x}_{r_0 p_x-1}$, $\overset{1_x}{h}$ and $\overset{2_x}{h}$ being defined by (6.5) and (6.12), has a set of $\underset{x}{\alpha}$ minimal numbers all zero, a second set of $\underset{x}{\gamma}$ minimal numbers all zero, and a non-singular square core of $\underset{x}{\beta}$ rows and columns whose elementary divisors are all linear, the most frequently repeated of which

[57] See (6.18), where primed rather than unprimed coordinates have been used.

is repeated $\beta - A - 1$ times, then condition A for H is satisfied but condition $A + 1$ is not.

We can write Lemmas VI.5 and VI.6 together as

THEOREM VI A 2. If $m = 2$, then the statements "Condition A for H is satisfied but condition $A + 1$ is not" and "The affinors h_1 and h_2 defined by (6.5) and (6.12) are equivalent to a non-singular pair of bilinear forms in $\beta + \beta$ variables whose elementary divisors are all linear, the most frequently repeated elementary divisor being repeated $\beta - A - 1$ times" are equivalent statements.

3. *The Case where* $m \geq 3$. We were able to give algebraic equivalents for condition A for H when m was 1 or 2, and of course we would like to do the same for $m \geq 3$. To do this, however, we need some knowledge of the invariants describing sets of three or more matrices. This subject in matrix algebra seems to be more difficult than the theory of the invariants of pairs of matrices, and the theory is far from complete.[48] For this reason, we are unable to complete the algebraic expression of our conditions at present.

B. **The Geometric Expression of the Conditions.** We can express the conditions on our curvature affinors $H_{r_0 r_1 \cdots r_{s-1} r_s}$ in more geometric language if we choose to, and we shall do that in this section.

Let us consider first the curvature affinor $H_{r_0 r_1}$. We have defined for it the vectors $u^{r_0}_A$, $A = 1, \cdots, \alpha_1$; $y^{r_0}_C$, $C = 1, \cdots, \gamma_1$; $v^{r_0}_B$, $B = 1, \cdots, \beta_1$; $x^{r_0}_B$; $\alpha_1 = \gamma_1$, with the following properties:

$$u^{r_0}_A H_{r_0 r_1} = 0, \tag{6.28}$$

$$y^{r_0}_C H_{r_0 r_1} = 0, \tag{6.29}$$

[48] The only literature known to the author on the subject of the invariants describing sets of three or more matrices is the following: S. Kantor, Sitzb. Akad. Munch., 27, 1897. S. Kantor, Monatsh. f Math u Phys, 11, 1900. G. E. F. Sherwood, Thesis, Chicago, 1922.

$$\overset{1}{v}{}^{r_0}_{B_1}\overset{1}{z}{}^{p_0}_{B_2}\overset{1}{H}_{r_0p_0}\cdots{}^{r_1} \begin{cases} = 0, & B_1 \neq B_2 \\ \neq 0, & B_1 = B_2. \end{cases} \tag{6.30}$$

We note that in this case all the vectors defined are vectors of the V_m.

We know that two vectors of the V_m, say X^{r_0} and Y^{p_0}, are conjugate vectors if and only if

$$X^{r_0}Y^{p_0}\overset{1}{H}_{r_0p_0}\cdots{}^{r_1} = 0.^{59} \tag{6.31}$$

Therefore (6.28) and (6.29) tell us that the vectors $\overset{1}{u}{}^{r_0}_{A}$, $\overset{1}{y}{}^{p_0}_{C}$ are vectors conjugate to all vectors of the V_m. We could call them 'omniconjugate' vectors if we wanted to. (6.30) tells us that the sets of vectors $\overset{1}{v}{}^{r_0}_{B}$, $\overset{1}{z}{}^{p_0}_{B}$ can be put into a 1–1 correspondence such that each vector of one set is conjugate to each vector of the other set except for the one vector corresponding to it.

From another point of view, two vectors of the V_m, say X^{r_0} and Y^{r_0}, are conjugate vectors if and only if the parallel displacement[60] in the direction X^{r_0} of each vector of the first normal space is perpendicular to Y^{r_0}. This follows from the first definition of conjugate vectors when one considers (6.4) for $x = 1$.[61]

But this definition of conjugate vectors can be generalized in a natural way to include vectors not on the V_m. We shall say that a vector of the V_m, say T^{r_0}, is quasi-conjugate to a vector of the $(x-1)^{st}$ normal space, say $W^{p_{x-1}}$, if and only if the parallel displacements[60] in the direction T^{r_0} of each of the vectors of the x^{th} normal space is perpendicular to $W^{p_{x-1}}$.

By this definition and by (6.4), we can say that if T^{r_0} is quasi-conjugate to $W^{p_{x-1}}$, then

$$\begin{aligned} T^{r_0}W^{p_{x-1}}\overset{x}{H}_{r_0p_{x-1}}\cdots{}^{r_x} &= -\overset{x-1}{B}{}^{a}_{p_{x-1}}W^{p_{x-1}}\overset{0}{B}{}^{a}_{r_0}T^{r_0}(D_\mu i_a)\overset{t_x}{i}{}^{r_x}_{t_x} \\ &= -W^{a}(T^{a}D_\mu i_a)\overset{t_x}{i}{}^{r_x}_{t_x} = 0. \end{aligned} \tag{6.32}$$

Working back, we can also say that if $T^{r_0}W^{p_{x-1}}\overset{x}{H}_{r_0p_{x-1}}\cdots{}^{r_x}$ vanishes, then by (6.4) $-W^{a}(T^{a}D_\mu i_a)\overset{t_x}{i}{}^{r_x}_{t_x}$ vanishes, and since the vectors $\overset{t_x}{i}{}^{r_x}_{t_x}$ are inde-

[59] II, pg. 89.

[60] Parallel displacements in the sense of Levi-Civita.

[61] Compare II, pg. 69, where the case of $V_{n-1} \subset V_n$ is treated.

pendent vectors, this means that

$$W^{s}(T^{\alpha}\overset{t_x}{D_{\mu}i_{s}}) = 0. \tag{6.33}$$

But (6.33) tells us that the directions $T^{\alpha\rho}$ and $W^{p_{x-1}}$ are quasi-conjugate. Therefore, a necessary and sufficient condition that the directions $T^{\alpha\rho}$ and $W^{p_{x-1}}$ be quasi-conjugate is that $T^{\alpha\rho}W^{p_{x-1}}\overset{x}{H}_{r\rho p_{x-1}}{}^{r_x}$ shall vanish.

Now we can see what condition 0 for $\overset{x}{H}$ means from this other point of view. The condition (2.1) means that there shall exist α_x vectors of the V_m, $\overset{x}{\underset{A}{u}}{}^{r\rho}$, which are quasi-conjugate to all the vectors of the $(x-1)^{st}$ normal space. The condition (2.3) means that there shall exist γ_x vectors of the $(x-1)^{st}$ normal space, $\overset{x}{\underset{C}{y}}{}^{p_{x-1}}$, which are quasi-conjugate to all vectors of the V_m. And finally, (2.2) requires that a set of β_x vectors of the V_m, $\overset{x}{\underset{B}{v}}{}^{r\rho}$, and a set of β_x vectors of the $(x-1)^{st}$ normal space, $\overset{x}{\underset{B}{z}}{}^{p_{x-1}}$, shall be in a 1–1 correspondence such that any given vector of one set shall be quasi-conjugate to each vector of the other set, except for the one vector corresponding to it.

Since we have already defined condition A for $\overset{x}{H}$ by adding conditions stated in geometric language to the definition of condition 0 for $\overset{x}{H}$, we have stated our conditions on the curvature affinors $\overset{x}{H}$ in geometric language.

VII. Conclusion

In this paper we have exhibited certain dependences among the Gauss-Codazzi-Ricci equations and have given sufficient conditions for these dependences. We have also stated these conditions in algebraic language for certain cases and in geometric language.

Of course, the general problem of the bending of surfaces in Riemannian manifolds is intimately connected with the statement of the Gauss-Codazzi-Ricci equations, and any work which simplifies the statement of these equations by showing that some are dependent on others under certain conditions is of indirect value at least. But more than that, the author is confident that the methods used in this paper are immedi-

ately applicable to the question of the rigidity of surfaces, if not to the central problem of bending itself. For instance, it seems that one can show that the x^{th} fundamental form of the V_m is determined by the first, second, third, $\cdots$, $(x-1)^{\text{st}}$ fundamental forms[62] under almost exactly the same conditions as those used in the theorems on the Gauss-Codazzi-Ricci equations. The author expects to discuss this subject in another paper.

THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

[62] The affinors L, the fundamental forms in the discussion of Schouten and Struik, are defined in II, pgs. 123, 124, 125.

A GENERAL TYPE OF LIE'S DIFFERENTIAL EQUATIONS

By Mario O. González

The Lie equations of the n^{th} order, that is, the equations which are invariant under a given group Uf, have the form[1]

$$\phi\left(u, u', \frac{du'}{du}, \cdots, \frac{d^{n-1}u'}{du^{n-1}}\right) = 0$$

where $u(x, y)$ is any invariant and $u'(x, y, y')$ is any first differential invariant of Uf; or the form[2]

$$\phi\left(v, \frac{dv}{du}, \frac{d^2v}{du^2}, \cdots, \frac{d^nv}{du^n}\right) = 0$$

where $v = v(x, y)$ and $u = u(x, y)$ are a pair of canonical variables for the group.

We begin by considering equations of the first order

$$\phi(u, u') = 0 \quad \text{or} \quad \phi\left(v, \frac{dv}{du}\right) = 0$$

and we next extend our consideration to higher order equations.

1. *A first generalization.*

Suppose we know that the differential equation

$$\frac{dy}{dx} = \varphi(x, y) \tag{1}$$

is invariant under the group

$$Uf = \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}$$

By hypothesis this group is not trivial with respect to the equation (1), i.e.

$$\xi\varphi(x, y) - \eta \not\equiv 0 \tag{2}$$

[1] Cohen, The Lie theory of one-parameter groups, 1931, p. 100.

[2] Franklin, The canonical form of a one-parameter group, Annals of Math., vol. 29, p. 115.

Now we try to find a differential equation invariant under

$$Uf = \omega(x)\left[\frac{\partial f}{\partial x} + \varphi(x, y)\frac{\partial f}{\partial y}\right] \tag{3}$$

Passing to the corresponding system of ordinary differential equations

$$\frac{dx}{\omega(x)} = \frac{dy}{\omega(x)\varphi(x, y)} = \frac{dY}{1}$$ [3]

we obtain from the first equation

$$\varphi(x, y)\,dx - dy = 0$$

We know that this equation has the integrating factor[4]

$$\frac{1}{\xi\varphi(x, y) - \eta}$$

hence the differential equation

$$\frac{\varphi(x, y)\,dx}{\xi\varphi(x, y) - \eta} - \frac{dy}{\xi\varphi(x, y) - \eta} = 0$$

is an exact differential $d\phi$, and we may adopt as first invariant

$$u = X = \phi$$

Taking as differential invariant

$$u' = \frac{dX}{dY} = \frac{\dfrac{\varphi(x, y)\,dx}{\xi\varphi(x, y) - \eta} - \dfrac{dy}{\xi\varphi(x, y) - \eta}}{\dfrac{dx}{\omega(x)}}$$

$$= \frac{\omega(x)}{\xi\varphi(x, y) - \eta}\left[\varphi(x, y) - \frac{dy}{dx}\right]$$

we have the differential equation invariant under (3)

$$u' = F(u)$$

or

$$\frac{dy}{dx} = \varphi(x, y) + \frac{\xi\varphi(x, y) - \eta}{\omega(x)}F(\phi) \tag{4}$$

[3] We denote any pair of canonical variables by X, Y.

[4] Lie-Scheffers, Vorlesungen über Differentialgleichungen, 1891, p. 97.

If we compare equation (4) with equation (1), we see that (4) has the additional term

$$\frac{\xi\varphi(x, y) - \eta}{\omega(x)} F(\phi)$$

which is not zero because $\xi\varphi(x, y) - \eta \not\equiv 0$, by (2), and F is an arbitrary function of ϕ. But if we suppose $F = 0$ then (4) reduces to (1). In others words, equation (4) is a generalization of equation (1).

This establishes our

THEOREM I. *Starting with a differential equation*

$$\frac{dy}{dx} = \varphi(x, y)$$

invariant under a known group of transformations

$$Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}$$

we can always write a new different type

$$\frac{dy}{dx} = \varphi(x, y) + \frac{\xi\varphi(x, y) - \eta}{\omega(x)} F(\phi)$$

more general than the type already known; ω and F are arbitrary (analytic) functions and ϕ is an invariant of the group

$$Uf = \omega(x)\left[\frac{\partial f}{\partial x} + \varphi(x, y)\frac{\partial f}{\partial y}\right]$$

2. *A correlative type.*

Finding the equation invariant under

$$Uf = \omega(y)\left[\frac{1}{\varphi(x, y)}\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right]$$

we get, similarly,

$$\frac{dy}{dx} = \frac{\varphi(x, y)\omega(y)}{\omega(y) + [\xi\varphi(x, y) - \eta]F(\phi)}$$

but obviously this type is not essentially different from (4).

3. *Relation between integrating factors.*

By Lie's theorem an integrating factor of equation (1) is

$$\mu = \frac{1}{\xi\varphi(x, y) - \eta}$$

and the general form for integrating factors of (1) will be

$$\rho = \mu G(\phi)$$

An integrating factor of equation (4) is

$$\mu' = \frac{1}{\omega(x)\left[\varphi(x, y) + \frac{1}{\mu\omega(x)} F(\phi)\right] - \omega(x)\varphi(x, y)} = \frac{\mu}{F(\phi)} = \mu G(\phi)^{5}$$

If μ' makes equation (4) an exact differential $d\phi'$, then the general form for integrating factors of (4) will be

$$\rho' = \mu' H(\phi') = \mu G(\phi) H(\phi')$$

This establishes

THEOREM II. *The general integrating factor of the equation*

$$\frac{dy}{dx} = \varphi(x, y)$$

is a particular integrating-factor of the equation

$$\frac{dy}{dx} = \varphi(x, y) + \frac{\xi\varphi(x, y) - \eta}{\omega(x)} F(\phi)$$

the general integrating factor of the latter being

$$\mu G(\phi) H(\phi')$$

4. *Further generalizations.*

Since equation (4) has the form

$$\frac{dy}{dx} = \psi(x, y)$$

and we know the group which leaves the differential equation invariant, we can generalize it in the same way. Performing the necessary opera-

[5] Evidently we can employ the same symbol G as before because it denotes an arbitrary function in both cases.

tions we find, introducing sub-indices,

$$\frac{dy}{dx} = \varphi(x, y) + \frac{\xi\varphi(x, y) - \eta}{\omega(x)} F_1(\phi_1)[1 + F_2(\phi_2)] \tag{5}$$

where ϕ_2 is an invariant of the group

$$Uf = \omega(x)\frac{\partial f}{\partial x} + \{\omega(x)\varphi(x, y) + [\xi\varphi(x, y) - \eta]F_1(\phi_1)\}\frac{\partial f}{\partial y}$$

any transformation of this group leaving (5) unaltered.

It is easy to prove, by mathematical induction, that after n successive generalizations we must find the equation

$$\frac{dy}{dx} = \varphi(x, y) + \frac{\xi\varphi(x, y) - \eta}{\omega(x)} F_1(\phi_1)[1 + F_2(\phi_2) + F_2(\phi_2)F_3(\phi_3) + \cdots + F_2(\phi_2)F_3(\phi_3) \cdots F_n(\phi_n)]$$

Often the invariants $\phi_2, \phi_3, \cdots, \phi_n$ have complicated expressions.

This is our general type of differential equations of the first order.

This enables us to enunciate the

THEOREM III. *From each type of differential equation of the first order arise infinitely many different types of differential equations of the same order.*

5. *Some applications.*

I. Prof. Franklin has given in his paper the following remarkable general type of differential equation

$$B(x) - \frac{A(x)}{C(y)}y' = F\left[\int \frac{B(x)}{A(x)}dx - \int \frac{dy}{C(y)}\right] \tag{6}$$

This type is by no means artificial; it appears naturally as a generalization of the equation

$$\frac{dy}{dx} = \varphi(x)\psi(y)$$

which is invariant under

$$Uf = \psi(y)\frac{\partial f}{\partial y}$$

Evidently

$$\xi\varphi(x, y) - \eta = -\psi(y)$$

$$\phi = \int \varphi(x)\,dx - \int \frac{dy}{\psi(y)}$$

then, by (4), we have

$$\frac{dy}{dx} = \psi(y)\left\{\varphi(x) - \frac{1}{\omega(x)} F\left[\int \varphi(x)\,dx - \int \frac{dy}{\psi(y)}\right]\right\} \tag{7}$$

Equations (6) and (7) are identical taking

$$\varphi(x) = \frac{B(x)}{A(x)}, \qquad \omega(x) = A(x), \qquad \psi(y) = C(y)$$

As Prof. Franklin shows, many types of first order differential equations considered in books on the subject are merely particular cases of (6). Also we can derive from (6) or (7) interesting new special types. For instance, put

$$\varphi(x) = \frac{\omega'(x)}{\omega(x)}, \qquad \psi(y) = -y$$

then

$$\frac{dy}{dx} = -y\frac{\omega'(x)}{\omega(x)} - \frac{y}{\omega(x)} F[\log \omega(x) + \log y]$$

or

$$\frac{dy}{dx} = -y\frac{\omega'(x)}{\omega(x)} + y^2\Phi(y\omega(x))$$

This equation is invariant under

$$Uf = \omega(x)\frac{\partial f}{\partial x} - y\omega'(x)\frac{\partial f}{\partial y}$$

EXAMPLE. Let $\omega(x) = e^x$ $\Phi(ye^x) = \log(ye^x) = x + \log y$ then

$$\frac{dy}{dx} = y(xy + y\log y - 1)$$

It has the integral

$$e^{-x} + li\left(\frac{e^{-x}}{y}\right) = c$$

II. Starting with the homogeneous equation

$$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right) = \varphi(v) \qquad (\varphi(v) \neq v)$$

we have as a first generalization

$$\frac{dy}{dx} = \varphi(v) + \frac{x\varphi(v) - y}{\omega(x)} F\left[\log x - \int \frac{dv}{\varphi(v) - v}\right]$$ [6]

III. Taking the linear differential equation in the form

$$\frac{dy}{dx} = \frac{\varphi'(x)}{\varphi(x)} y + \psi(x)$$

we have the generalization

$$\frac{dy}{dx} = \frac{\varphi'(x)}{\varphi(x)} y + \psi(x) + \frac{\varphi(x)}{\omega(x)} F\left[\frac{y}{\varphi(x)} - \int \frac{\psi(x)}{\varphi(x)} dx\right]$$

IV. As another example, take

$$\frac{dy}{dx} = \varphi(ax + by + c) = \varphi(v)$$

The generalized type is

$$\frac{dy}{dx} = \varphi(v) + \frac{b\varphi(v) + a}{\omega(x)} F\left[x - \int \frac{dv}{b\varphi(v) + a}\right]$$

6. *Higher order equations.*

By a remarkable Lie's theorem, as a second differential invariant u'' for an extended group of transformations we can take

$$\frac{du'}{du}$$

and, in general,

$$u^{(n)} = \frac{d^{n-1}u'}{du^{n-1}}$$

In the following we shall assume $\omega(x) = 1$ for simplicity.

As we have taken (§1)

$$u' = \frac{dX}{dY}$$

being

$$dX = du, \qquad dY = dx$$

[6] M. O. González, Una generalización de la ecuación diferencial homogénea, Bol. Mat., Oct. 1938.

we get

$$u' = \frac{du}{dx}$$

hence

$$u'' = \frac{du'}{du} = \frac{du'}{dx}\frac{dx}{du} = \frac{d^2u}{dx^2}\frac{1}{u'}$$

and the equation of second order can be written

$$\frac{d^2u}{dx^2}\frac{1}{u'} = f(u, u')$$

or

$$\frac{d^2u}{dx^2} = F(u, u')$$

In a similar way we have for the equation of n^{th} order

$$\frac{d^nu}{dx^n} = F(u, u', u'', \cdots, u^{(n-1)})$$

7. *Applications.*

I. In Franklin's equation the invariant is

$$u = \int \varphi(x)\,dx - \int \frac{dy}{\psi(y)}$$

hence

$$\frac{du}{dx} = \varphi(x) - \frac{y'}{\psi(y)}$$

$$\frac{d^2u}{dx^2} = \varphi'(x) - \frac{y''}{\psi(y)} + \left[\frac{y'}{\psi(y)}\right]^2 \psi'(y)$$

and the corresponding equation of second order is

$$y'' = \varphi'(x)\psi(y) + \frac{\psi'(y)}{\psi(y)}\,y'^2 + \psi(y)F\left[\int \varphi(x)\,dx - \int \frac{dy}{\psi(y)}, \varphi(x) - \frac{y'}{\psi(y)}\right]$$

II. In the linear equation we have

$$u = \frac{y}{\varphi(x)} - \int \frac{\psi(x)}{\varphi(x)}\,dx$$

Then we obtain

$$y'' = 2y' \frac{\varphi'(x)}{\varphi(x)} + y \frac{\varphi''(x)}{\varphi(x)} - 2y \left[\frac{\varphi'(x)}{\varphi(x)}\right]^2 + \psi'(x) - \psi(x) \frac{\varphi'(x)}{\varphi(x)}$$
$$+ \varphi(x) F\left[\frac{y}{\varphi(x)} - \int \frac{\psi(x)}{\varphi(x)} dx,\ \frac{y'}{\varphi(x)} - y \frac{\varphi'(x)}{\varphi(x)^2} - \frac{\psi(x)}{\varphi(x)}\right]$$

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Massachusetts Institute of Technology

AN INTEGRAL INEQUALITY OF THE PHRAGMEN LINDELOF TYPE

By Norman Levinson

THEOREM I. *Let $H(z)$ be analytic in a sector $|\theta| \leqq \pi/2\rho$, and let*

$$\int_0^\infty |H(re^{\pm\pi i/2\rho})|^p\, dr \leqq M$$

for some p, $0 < p < \infty$. Let

$$\overline{\lim_{r\to\infty}} \frac{\log |H(re^{i\theta})|}{r^\rho} \leqq a, \qquad (|\theta| \leqq \pi/2\rho).$$

Then

$$\int_0^\infty |H(re^{i\theta})e^{-ar^\rho \cos\rho\theta}|^p\, dr \leqq M, \qquad (|\theta| \leqq \pi/2\rho).$$

By mapping the sector onto a half-plane, this theorem can be reduced to the case where $\rho = 1$. That is Theorem I is true if the following particular case is proved.

THEOREM II. *Let $H(z)$ be analytic in the upper half-plane $y \geqq 0$, $(z = x + iy)$. Let*

$$\text{Max}\left\{\int_0^\infty |H(x)|^p\, dx, \int_{-\infty}^0 |H(x)|^p\, dx\right\} = M \tag{1.01}$$

for some p, $0 < p < \infty$. Let

$$\overline{\lim_{|z|\to\infty}} \frac{\log |H(z)|}{|z|} \leqq a, \qquad (0 \leqq \text{am } z \leqq \pi). \tag{1.02}$$

Then for $0 \leqq \theta \leqq \pi$,

$$\int_0^\infty |H(re^{i\theta})e^{-ar\sin\theta}|^p\, dr \leqq M. \tag{1.03}$$

2. In proving Theorem II we require several lemmas.

Lemma 2.1. *If the zeros of $H(z)$ be denoted by $\{z_n\}$ where $z_n = r_n e^{i\theta_n}$, then*

$$\sum_1^\infty \frac{\sin\theta_n}{r_n} < \infty. \tag{2.01}$$

Proof of Lemma 2.1. Applying a theorem of Carleman[1] to $H(z)$ we have

$$\sum_{r_n \leq R} \frac{\sin \theta_n}{r_n}\left(1 - \frac{r_n^2}{R^2}\right) \leqq \frac{1}{\pi R}\int_0^\pi \log |H(Re^{i\theta})| \sin \theta \, d\theta + \frac{1}{2\pi}\left(\int_{-R}^{-1} + \int_1^R\right)\left(\frac{1}{x^2} - \frac{1}{R^2}\right)\log |H(x)|\, dx + A \tag{2.02}$$

for some constant A. Using (1.02)

$$\sum_{r_n \leq R} \frac{\sin \theta_n}{r_n}\left(1 - \frac{r_n^2}{R^2}\right) \leqq \frac{1}{2\pi p}\left(\int_{-R}^{-1} + \int_1^R\right)\left(\frac{1}{x^2} - \frac{1}{R^2}\right)\log |H(x)|^p\, dx + A_1 \tag{2.03}$$

for some A_1. But $\log |H(x)|^p \leqq |H(x)|^p$. Thus using (1.01), (2.03) gives

$$\sum_{r_n \leq \frac{1}{2}R} \frac{\sin \theta_n}{r_n}\left(1 - \frac{r_n^2}{R^2}\right) \leqq A_2$$

for some A_2. Since $r_n^2/R^2 \leqq \frac{1}{2}$ if $r_n \leqq \frac{1}{2}R$,

$$\sum_{r_n \leq \frac{1}{2}R} \frac{\sin \theta_n}{r_n} \leqq 2A_2.$$

Letting $R \to \infty$ this gives (2.01).

Lemma 2.2. *There exists a function $H_1(z)$ analytic for $y \geqq 0$ such that $H_1(z)$ has no zeros for $y > 0$ and*

$$|H_1(x)| = |H(x)|, \tag{2.04}$$

$$|H_1(z)| \geqq |H(z)|, \qquad (y > 0). \tag{2.05}$$

Also for some A

$$H_1(z) = O(e^{A|z|}), \qquad (y \geqq 0), \tag{2.06}$$

and

$$\lim_{y \to \infty} \frac{\log |H_1(iy)|}{y} \leqq a. \tag{2.07}$$

Proof of Lemma 2.2. Let

$$F(z) = \prod_1^\infty \left(\frac{1 - \frac{z}{\bar{z}_n}}{1 - \frac{z}{z_n}}\right). \tag{2.08}$$

[1] Titchmarsh, Theory of Functions, p. 130.

By (2.01) the product exists. Clearly

$$|F(z)| = 1$$

and

$$|F(z)| \geqq 1, \qquad (y \geqq 0).$$

Thus if

$$H_1(z) = F(z)H(z) \tag{2.09}$$

then (2.04) and (2.05) are true and $H_1(z)$ has no zeros for $y > 0$. By an equality for the half-plane[2] analogous to the Poisson theorem for the circle, it follows that

$$\begin{aligned}\frac{\log|H_1(re^{i\theta})|}{r\sin\theta} = \frac{1}{2\pi}\int_{-R}^{R}\log|H_1(x)|\left[\frac{1}{x^2-2xr\cos\theta+r^2}\right.\\ \left.-\frac{R^2}{R^4-2rxR^2\cos\theta+r^2x^2}\right]dx\\ +\frac{R}{\pi}\int_0^{\pi}\log|H_1(Re^{i\phi})|\frac{\sin\phi(R^2-r^2)}{|R^2e^{2i\phi}-2rR\cos\theta e^{i\phi}+r^2|^2}d\phi.\end{aligned} \tag{2.10}$$

Since the bracketed term in the first integral on the right side of (2.10) is positive, the log terms on the right of (2.10) can be changed to $\log^+$. Doing this and then discarding the negative part of the bracketed term gives

$$\begin{aligned}\frac{\log|H_1(re^{i\theta})|}{r\sin\theta} \leqq \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\log^+|H_1(x)|}{x^2-2xr\cos\theta+r^2}dx\\ +\overline{\lim_{R\to\infty}}\frac{1}{\pi R}\int_0^{\pi}\log^+|H_1(Re^{i\phi})|\sin\phi\,d\phi.\end{aligned} \tag{2.11}$$

Since $|H_1(x)| = |H(x)|$ and $H_1(z) = H(z)F(z)$, (2.11) becomes

$$\begin{aligned}\frac{\log|H_1(re^{i\theta})|}{r\sin\theta} \leqq \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\log^+|H(x)|}{x^2-2xr\cos\theta+r^2}dx\\ +\overline{\lim_{R\to\infty}}\frac{1}{\pi R}\int_0^{\pi}\log^+|H(Re^{i\phi})|\sin\phi\,d\phi\\ +\overline{\lim_{R\to\infty}}\frac{1}{\pi R}\sum_{n=1}^{\infty}\int_0^{\pi}\log^+\left|\frac{z_n-Re^{i\phi}}{z_n-Re^{i\phi}}\right|\sin\phi\,d\phi.\end{aligned} \tag{2.12}$$

[2] See for example, Levinson, *On the Closure of* $e^{\Omega_n z}$ *and Integral Functions.* Proc. Camb. Phil. Soc., Vol. XXXI, p. 338, Lemma 1.

Using $p \log^+ |H(x)| \leqq |H(x)|^p$ gives for $0 \leqq \theta \leqq \frac{1}{2}\pi$,

$$\left(\int_{-\infty}^{\frac{1}{2}r} + \int_{2r}^{\infty}\right) \frac{\log^+ |H(x)|}{x^2 - 2xr\cos\theta + r^2}\,dx \leqq \frac{1}{p}\left(\int_{-\infty}^{\frac{1}{2}r} + \int_{2r}^{\infty}\right) |H(x)|^p\,dx.$$

Also since $|H(x)| \leqq e^{(a+1)|x|+c}$,

$$\begin{aligned}\int_{\frac{1}{2}r}^{2r} \frac{\log^+ |H(x)|}{x^2 - 2xr\cos\theta + r^2}\,dx &\leqq \int_{\frac{1}{2}r}^{2r} \frac{(a+1)x + c}{x^2 - 2xr\cos\theta + r^2}\,dx \\ &\leqq [2(a+1)r + c]\int_{\frac{1}{2}r}^{2r} \frac{dx}{(x-r)^2 + r^2\sin^2\frac{1}{2}\theta} \\ &\leqq \frac{2(a+1)r + c}{r\sin\frac{1}{2}\theta}\,\pi.\end{aligned}$$

Using these results in (2.12)

$$\begin{aligned}\frac{\log |H_1(re^{i\theta})|}{r\sin\theta} \leqq \frac{A}{\sin\frac{1}{2}\theta} &+ \overline{\lim_{R\to\infty}} \int_0^{\pi} \log^+ |H(Re^{i\phi})| \sin\phi\,d\phi \\ &+ \overline{\lim_{R\to\infty}} \frac{1}{\pi R} \sum_1^{\infty} \int_0^{\pi} \log^+ \left|\frac{Re^{i\phi} - \bar{z}_n}{Re^{i\phi} - z_n}\right| \sin\phi\,d\phi,\end{aligned}$$

for some A and for $0 \leqq \theta \leqq \frac{1}{2}\pi$. Using (1.02) and noticing that $\sin\theta/\sin\frac{1}{2}\theta \leqq 2$ the above inequality becomes

$$\log |H_1(re^{i\theta})| \leqq A_1 r + r\,\overline{\lim_{R\to\infty}} \frac{1}{\pi R} \sum_1^{\infty} \int_0^{\pi} \log^+ \left|\frac{Re^{i\phi} - \bar{z}_n}{Re^{i\phi} - z_n}\right| \sin\phi\,d\phi \tag{2.13}$$

for some A_1. (2.13) holds for $\frac{1}{2}\pi < \theta \leqq \pi$ in much the same way as for $0 \leqq \theta \leqq \frac{1}{2}\pi$.

By taking

$$\frac{1}{2\pi i} \int \log\left(\frac{s - \bar{z}_n}{s - z_n}\right)\left(\frac{1}{s^2} - \frac{1}{R^2}\right) ds$$

around a semi-circle of radius R in the upper half-plane and then taking the real part it follows easily that

$$\frac{1}{\pi R} \int_0^{\pi} \log \left|\frac{Re^{i\phi} - \bar{z}_n}{Re^{i\phi} - z_n}\right| \sin\phi\,d\phi = \begin{cases} \dfrac{\sin\theta_n}{r_n}\left(1 - \dfrac{r_n^2}{R^2}\right), & (r_n < R). \\ 0, & (r_n \geqq R). \end{cases} \tag{2.14}$$

Since z_n is in the upper half-plane it also follows that log in (2.14) can be replaced by $\log^+$. Thus (2.13) becomes

$$\log |H_1(re^{i\theta})| \leqq A_1 r + r\,\overline{\lim_{R\to\infty}} \sum_{r_n \leqq R} \frac{\sin\theta_n}{r_n}.$$

By (2.01) it follows that

$$\log |H_1(re^{i\theta})| \leqq A_3 r.$$

This gives (2.06).

Clearly

$$|H_1(iy)| = |H(iy)| \prod_1^\infty \left| \frac{\bar{z}_n - iy}{z_n - iy} \right|.$$

Thus if $z_n = x_n + iy_n$,

$$\frac{\log |H_1(iy)|}{y} = \frac{\log |H(iy)|}{y} + \frac{1}{2y} \sum_1^\infty \log \frac{x_n^2 + (y + y_n)^2}{x_n^2 + (y - y_n)^2}.$$

Using (1.02)

$$\text{(2.15)} \quad \varlimsup_{y \to \infty} \frac{\log |H_1(iy)|}{y} \leqq a + \varlimsup_{y \to \infty} \frac{1}{2y} \sum_1^\infty \log \left(1 + \frac{4yy_n}{x_n^2 + (y - y_n)^2}\right).$$

Clearly

$$\sum_{|z_n| \geq 2y} \log \left(1 + \frac{4yy_n}{x_n^2 + (y - y_n)^2}\right) \leqq \sum_{|z_n| \geq 2y} \log \left(1 + \frac{4yy_n}{x_n^2 + y_n^2}\right)$$

$$\leqq \sum_{|z_n| \geq 2y} \frac{4yy_n}{x_n^2 + y_n^2} = \sum_{r_n \geq 2y} \frac{4y \sin \theta_n}{r_n}.$$

Using this and (2.01) in (2.15) it follows that

$$\text{(2.16)} \quad \varlimsup_{y \to \infty} \frac{\log |H_1(iy)|}{y} \leqq a + \varlimsup_{y \to \infty} \frac{1}{2y} \sum_{r_n < 2y} \log \left(1 + \frac{4yy_n}{x_n^2 + (y - y_n)^2}\right).$$

Again

$$\varlimsup_{y \to \infty} \frac{1}{2y} \sum_{r_n \leq 1y} \log \left(1 + \frac{4yy_n}{x_n^2 + (y - y_n)^2}\right) \leqq \varlimsup_{y \to \infty} \frac{1}{2y} \sum_{r_n \leq 1y} \log \left(1 + \frac{16yy_n}{y^2}\right)$$

$$\leqq \varlimsup_{y \to \infty} \frac{8}{y^2} \sum_{r_n \leq 1y} r_n \sin \theta_n$$

$$\leqq 8 \varlimsup_{y \to \infty} \sum_{r_n \leq 1y} \frac{\sin \theta_n}{r_n} \left(\frac{r_n^2}{y^2}\right).$$

Again by (2.01) it follows easily that the above limit must be zero.

Thus (2.16) becomes

$$\varlimsup_{y\to\infty} \frac{\log|H_1(iy)|}{y} \leqq a + \varlimsup_{y\to\infty} \frac{1}{2y} \sum_{\frac{1}{2}y<r_n<2y} \log\left(1 + \frac{4yy_n}{x_n^2 + (y-y_n)^2}\right). \tag{2.17}$$

Unless $\frac{1}{4}\pi < \theta_n < \frac{3}{4}\pi$,

$$\log\left(1 + \frac{4yy_n}{x_n^2 + (y-y_n)^2}\right) \leqq \log\left(1 + \frac{4yy_n}{x_n^2}\right) \leqq \log\left(1 + \frac{8yy_n}{r_n^2}\right) \leqq 8y\frac{\sin\theta_n}{r_n}.$$

Using this and (2.01) in (2.17) and denoting

$$\sum_{\substack{\frac{1}{2}y<r_n<2y \\ \frac{1}{4}\pi<\theta_n<\frac{1}{4}\pi}}$$

by $\sum'$, it follows that

$$\varlimsup_{y\to\infty} \frac{\log|H_1(iy)|}{y} \leqq a + \varlimsup_{y\to\infty} \frac{1}{2y} \sum{}' \log\left(1 + \frac{4yy_n}{x_n^2 + (y-y_n)^2}\right) \leqq a + \varlimsup_{y\to\infty} \frac{1}{y} \sum{}' \log\left|\frac{y+y_n}{y-y_n}\right|. \tag{2.19}$$

Since for $\frac{1}{4}\pi < \theta_n < \frac{3}{4}\pi$, $\sin\theta_n > 1/\sqrt{2}$,

$$\sum_{\frac{1}{4}\pi<\theta_n<\frac{1}{4}\pi} \frac{1}{y_n} \leqq 2 \sum_{\frac{1}{4}\pi<\theta_n<\frac{1}{4}\pi} \frac{\sin\theta_n}{r_n} < 2\sum_1^\infty \frac{\sin\theta_n}{r_n}.$$

Thus $\{y_n\}$ for $\frac{1}{4}\pi < \theta_n < \frac{3}{4}\pi$ are not spaced very densely and obviously there are points in any interval $(2^k, 2^{k+1})$ for large k where $\min|y - y_n| \geqq 1$, $(\frac{1}{4}\pi < \theta_n < \frac{3}{4}\pi)$. Also for some intervals $(2^k, 2^{k+1})$ the number of z_n, $(2^k/10 < |z_n| < (10)2^{k+1}$, $\frac{1}{4}\pi < \theta_n < \frac{3}{4}\pi)$, are

$$o\left(\frac{2^k}{k}\right) \quad \text{or} \quad o\left(\frac{2^k}{\log 2^k}\right)$$

for otherwise

$$\sum_{\frac{1}{4}\pi<\theta_n<\frac{1}{4}\pi} \frac{1}{y_n} = \infty.$$

Thus for some y in these intervals

$$\sum' \log \left| \frac{y + y_n}{y - y_n} \right| \leqq \sum' \log |y + y_n| \leqq \sum' \log (3y)$$
$$= o\left(\frac{y}{\log y}\right) \log 3y = o(y).$$

Using this in (2.18)

$$\varlimsup_{y \to \infty} \frac{\log |H_1(iy)|}{y} \leqq a.$$

This completed the proof of this lemma.

3. *Proof of Theorem II.* Applying Carleman's theorem, (2.02), to using (2.04) and (2.06), it follows easily that

$$\int_{-\infty}^{\infty} \log^- |H_1(x)| \frac{dx}{x^2} > -\infty, \tag{3.01}$$

and

$$\varlimsup_{R \to \infty} \frac{1}{R} \int_0^{\pi} \log^- |H_1(Re^{i\theta})| \sin \theta \, d\theta > -\infty. \tag{3.02}$$

Using (2.10) it follows easily that

$$\begin{aligned} \frac{\log |H_1(re^{i\theta})|}{r \sin \theta} = \frac{1}{2\pi} \int_{-\infty}^{\infty} & \frac{\log |H_1(x)|}{x^2 - 2xr \cos \theta + r^2} dx \\ & + O\left(\varlimsup_{R \to \infty} \frac{1}{R^2} \int_{-R}^{R} |\log |H_1(x)|| \, dx \right) \\ & + \varlimsup_{R \to \infty} \frac{1}{\pi R} \int_0^{\pi} \log |H_1(Re^{i\theta})| \sin \theta \, d\theta \\ & + O\left(\varlimsup_{R \to \infty} \frac{1}{R^2} \int_0^{\pi} |\log |H_1(Re^{i\theta})|| \sin \theta \, d\theta \right). \end{aligned} \tag{3.03}$$

By (2.06) and (3.02) the last term in the right is zero. Also

$$\begin{aligned} \frac{1}{R^2} \int_{-R}^{R} |\log |H_1(x)|| \, dx \leqq & \frac{1}{R^2} \int_{-A}^{A} |\log |H_1(x)|| \, dx \\ & + \left(\int_{-R}^{-A} + \int_{A}^{R} \right) |\log |H_1(x)|| \frac{dx}{x^2} \end{aligned}$$

Letting $R \to \infty$,

$$\varlimsup_{R \to \infty} \frac{1}{R^2} \int_{-R}^{R} |\log |H_1(x)|| \, dx \leqq \left(\int_{-\infty}^{-A} + \int_{A}^{\infty} \right) |\log |H_1(x)|| \frac{dx}{x^2}.$$

Since A is arbitrary, the left side above must be zero. Thus (3.03) becomes

$$\frac{\log|H_1(re^{i\theta})|}{r\sin\theta} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\log|H_1(x)|}{x^2-2xr\cos\theta+r^2}dx + \overline{\lim_{R\to\infty}}\frac{1}{\pi R}\int_0^{\pi}\log|H_1(Re^{i\theta})|\sin\theta\,d\theta.$$

But this implies that $\overline{\lim}$ exists and is equal to some constant B. Thus

$$\frac{\log|H_1(re^{i\theta})|}{r\sin\theta} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\log|H_1(x)|}{x^2-2xr\cos\theta+r^2}dx + B. \tag{3.04}$$

In particular let $\theta = \frac{1}{2}\pi$. Then

$$\frac{\log|H_1(iy)|}{y} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\log|H_1(x)|}{x^2+y^2}dx + B.$$

Letting $y \to \infty$ it follows that

$$\lim_{y\to\infty}\frac{\log|H_1(iy)|}{y} = B.$$

This and (2.07) imply that

$$B \leqq a. \tag{3.05}$$

(3.04) can be written as

$$\log|H_1(z)| = \frac{y}{2\pi}\int_{-\infty}^{\infty}\frac{\log|H_1(\xi)|}{\xi^2-2x\xi+x^2+y^2}d\xi + By. \tag{3.06}$$

Let

$$\phi_N(x) = \begin{cases} |H_1(x)|, & (|H_1(x)| \leqq N), \\ N, & (|H_1(x)| > N). \end{cases} \tag{3.07}$$

Then there exists the harmonic function

$$\frac{y}{2\pi}\int_{-\infty}^{\infty}\frac{\log\phi_N(\xi)}{(x-\xi)^2+y^2}d\xi + By$$

in the upper half-plane. Thus there exists an analytic function in the upper half-plane, $G_N(z)$ such that

$$\log|G_N(z)| = \frac{y}{2\pi}\int_{-\infty}^{\infty}\frac{\log\phi_N(\xi)}{(x-\xi)^2+y^2}d\xi + By. \tag{3.08}$$

Clearly

$$| G_N(x) | \leqq N,$$

and

$$\log | G_N(z) | = O(| z |)$$

and

$$\overline{\lim_{y \to \infty}} \frac{\log | G_N(iy) |}{y} \leqq B.$$

Thus by the well known theorem of Phragmen-Lindelof

$$| G_N(z) e^{i(B+\epsilon)z} | \leqq N$$

in the upper half-plane. Since this is true for any $\epsilon > 0$,

$$| G_N(z) e^{iBz} | \leqq N \tag{3.09}$$

in the upper half-plane. Let

$$J_N(z) = G_N(z) e^{iBz - \delta\sqrt{(-iz)}}, \tag{3.10}$$

and for some θ, $0 < \theta < \pi$, let

$$f(t) = \frac{\overline{J_N(te^{i\theta})}}{| J_N(te^{i\theta}) |}.$$

Let

$$K_N(z) = z \int_0^\infty [f(t) J_N(zt)]^p \, dt.$$

Clearly if $0 \leqq \operatorname{am} z \leqq \pi$,

$$| K_N(z) | \leqq | z | \int_0^\infty | J_N(zt) |^p \, dt = \int_0^\infty \left| J_N\left(\frac{zt}{|z|}\right) \right|^p dt$$

$$\leqq N^p \int_0^\infty | e^{-\delta\sqrt{(-izt/|z|)}} |^p \, dt.$$

That is $K_N(z)$ is bounded in the upper half-plane. Again by Phragmen Lindelof

$$| K_N(z) | \leqq \operatorname*{Max}_{-\infty < x < \infty} | K_N(x) |.$$

But

$$\operatorname*{Max}_{-\infty < x < \infty} | K_N(x) | \leqq \operatorname{Max}\left(\int_0^\infty | J_N(t) |^p \, dt, \int_0^\infty | J_N(-t) |^p \, dt \right).$$

Thus

$$|K_N(z)| \leqq \text{Max}\left(\int_0^\infty |J_N(t)|^p\,dt,\ \int_0^\infty |J_N(-t)|^p\,dt\right).$$

Let $z = e^{i\theta}$. Then

$$K_N(e^{i\theta}) = \int_0^\infty |J_N(te^{i\theta})|^p e^{i\theta}\,dt.$$

Thus

$$\int_0^\infty |J_N(te^{i\theta})|^p\,dt \leqq \text{Max}\left(\int_0^\infty |J_N(t)|^p\,dt,\ \int_0^\infty |J_N(-t)|^p\,dt\right).$$

By (3.10) this gives

$$\int_0^\infty |J_N(te^{i\theta})|^p\,dt \leqq \text{Max}\left(\int_0^\infty |G_N(x)|^p\,dx,\ \int_0^\infty |G_N(-x)|^p\,dx\right).$$

Let $\delta \to 0$ above. Then by (3.10)

$$\int_0^\infty |G_N(z)e^{iBz}|^p\,|dz| \leqq \text{Max}\left(\int_0^\infty |G_N(x)|^p\,dx,\ \int_0^\infty |G_N(-x)|^p\,dx\right).$$

Or since $|G_N(x)| \leqq |H_1(x)|$,

$$\int_0^\infty |G_N(re^{i\theta})e^{-Br\sin\theta}|^p\,dr \leqq M.$$

But by (3.06), (3.07), and (3.08)

$$\lim_{N\to\infty} |G_N(z)| = |H_1(z)|.$$

Thus

$$\int_0^\infty |H_1(re^{i\theta})e^{-Br\sin\theta}|^p\,dr \leqq M.$$

But $|H_1(z)| \geqq |H(z)|$ and $B \leqq a$. Thus

$$\int_0^\infty |H(re^{i\theta})e^{-ar\sin\theta}|^p\,dr \leqq M.$$

This completes the proof of Theorem II.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

A THEOREM ON MAXIMA AND MINIMA WITH AN APPLICATION TO DIFFERENTIAL EQUATIONS

By Louis M. Court

THEOREM: Let $\phi(q_1, \cdots, q_n)$ have continuous first[1] and finite second derivatives in some subregion **A** of its region of definition. Suppose further that ϕ can be maximized(minimized) subject to any one of an ∞^n aggregate of linear conditions; and that each point $Q = (q_1^{(P)}, \cdots, q_n^{(P)})$ of **A** has, and can have, *only* one condition $\sum_{i=1}^{n} p_i^{(Q)} q_i = m$ associated with it which renders ϕ a proper[2] maximum(minimum) at Q. Then, if $p_i = p_i(q_1, \cdots, q_n)$ $(i = 1, \cdots, n)$ are the maximizing(minimizing) relations and $q_i = q_i(p_1, \cdots, p_n)$ $(i = 1, \cdots, n)$ their solutions or inverses defined in some region **B** of the $(p_1, \cdots, p_n)$ space, the function $\psi(p_1, \cdots, p_n) \equiv \phi[q_1(p_1, \cdots, p_n), \cdots, q_n(p_1, \cdots, p_n)]$ will be properly minimized(maximized) at each point $P = (p_1^{(Q)}, \cdots, p_n^{(Q)})$ of **B** (whose points because of a prior assumption are in one-to-one correspondence with the points of **A**) subject to the constraint $\sum_{i=1}^{n} q_i^{(P)} p_i = m$, with the q's held constant now instead of the p's, by the identical relations $q_i = q_i(p_1, \cdots, p_n)$ or, what amounts to the same thing, their inverses $p_i = p_i(q_1, \cdots, q_n)$. It is plain that if Q and P are two corresponding points, $\sum_{i=1}^{n} p_i^{(Q)} q_i^{(P)} = m$.

PROOF: We will first give the demonstration for $n = 2$ because it is easily pictured, as on page 100.

The curves drawn inside the region **A** in the figure are the contour lines or curves of constant value of $\phi(q_1, q_2)$. Those inside **B** are the contour lines of

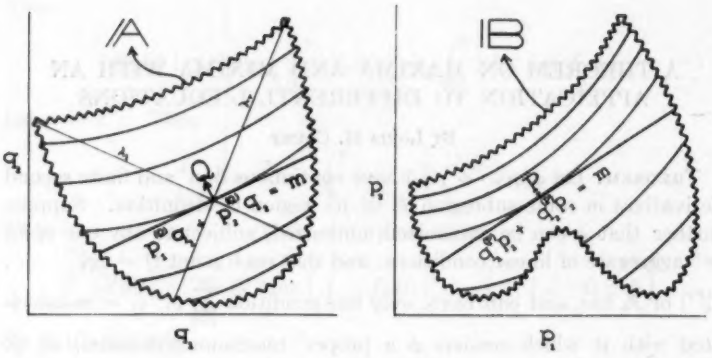
$$\phi[q_1(p_1, p_2), q_2(p_1, p_2)] \equiv \psi(p_1, p_2).$$

In effect the relation $\psi(p_1, p_2) \equiv \phi[q_1(p_1, p_2), q_2(p_1, p_2)]$ assigns to each point $P = (p_1^{(Q)}, p_2^{(Q)})$ within **B** the maximum(minimum) value that

Theorem established during the author's tenure of the Granville W. Garth Fellowship at Columbia University.

[1] The continuity of ϕ in **A** follows from the existence and continuity of its partial derivatives throughout the subregion.

[2] I.e. not a minimax or other point satisfying only the first order conditions, for an extreme.



$\phi(q_1, q_2)$ assumes on the line $p_1^{(Q)}q_1 + p_2^{(Q)}q_2 = m$ or the value of ϕ at the point $Q = (q_1^{(P)}, q_2^{(P)})$ at which this line is tangent to a contour curve.

Consider the line tangent to the contour curve of $\psi(p_1, p_2)$ through P. Because of the setup and the limitations imposed on **A**, and therefore on **B**, there is a one-to-one correspondence between the points of the segment of the line contained within **B** and a fraction of the pencil of lines through Q. The left limit of the fraction or portion is the ray λ_l which is the steepest line on the left tangent to a contour curve; and the right limit in the figure is λ_r.

In proving the theorem, three geometric lemmas are needed to implement this correspondence. First, the tangent to a contour line at the general point Q cannot cross the curve. For if it did, ϕ could be maximized(minimized) at Q subject to the constraint only on condition the function assumed a higher(lower) value along the particular curve than along any other contour curve cut by the tangent. In that case, however, ϕ would be maximized(minimized) at Q subject to the constraint represented by *any* line through the point—contrary to the explicit assumption of uniqueness in the hypothesis of the theorem.

Secondly, all lines of the fractional pencil through the general point Q, other than the tangent, cross the contour curve through Q and touch curves on one side of this curve. For, on assumption of the opposite, let α be a line tangent to a contour curve on one side and β on the other. Then, in accord with the first lemma, α and β fail to decussate the curves they touch. The curvature of one of these curves (say the one touched by α) is positive at the point of tangency while the curvature of the other is negative. Therefore, provided we assume the second deriva-

tives of ϕ to be continuous,[3] the curvature is a continuous function of position and there exists a continuum[4] of at least one dimension for which it vanishes. If this continuum is two dimensional—i.e. essentially an area—contour curve parts situate within it are straight lines since their curvature is everywhere zero. So that, ϕ cannot be properly maximized (minimized) subject to *any* linear constraint at *any* interior point of the continuum—contrary to the hypothesis of the theorem. Should the continuum be one dimensional—i.e. a continuous curve—and coincident with portions of one or more contour curves, the same contradiction applies. If nowhere coincident, tangents to contour lines at points of the continuum, other than end points, cross the curves they touch. This is so since the curvature changes sign in passing from one side of the continuum to the other. The decussation of the tangents, however, contradicts the first lemma and the second lemma is established.

Thirdly, for some small swath surrounding the curve through Q, ϕ assumes higher(lower) values on the side on which the lines of the fractional pencil, other than the tangent, touch contour curves than on the opposite. Otherwise the tangent to the particular curve would cut contour curves with greater(smaller) values than the value of ϕ at Q. And the function would be minimized(maximized) rather than maximized(minimized) at Q subject to the constraint represented by the tangent.

Hence, we have proven that *all* the lines of the fractional pencil (except $p_1^{(Q)}q_1 + p_2^{(Q)}q_2 = m$) are tangent to contour curves on one side of the curve through Q—that is to contour curves along which, for at least some small swath surrounding the curve through Q, ϕ assumes higher(lower) values than at Q. Hence, the values of ψ along the points of the tangent line through P ($q_1^{(P)}p_1 + q_2^{(P)}p_2 = m$), which are equal to these values of ϕ, are for at least some segment about P higher (lower) than the value of ψ at P. Hence, ψ is minimized(maximized) at P subject to $q_1^{(P)}p_1 + q_2^{(P)}p_2 = m$.

The extension to n dimensions is straightforward. We stop to consider the geometrical properties of the contour hypersurfaces of ϕ. No such hypersurface can be of intrinsic dimensionality r ($0 \leqq r < n - 1$).

[3] The proof for the general n, given later, does not call for continuity on the part of the second derivatives of ϕ. Use of curvature, as the function vanishing along a "continuum," necessitates such an assumption here.

[4] In this paper the term "continuum" will signify a perfect set lacking none, part, or all of its boundary.

For if it were, ϕ would assume a higher (lower) value W on the hypersurface than at any neighboring point. Otherwise, since ϕ is a continuous function of position, the point set separating points with values of ϕ superior to W from those inferior would have the dimensionality $n - 1 = r$. But, if W is a maximum(minimum), ϕ could be maximized (minimized) at any point of the hypersurface subject to the constraint represented by *any* hyperplane[5] through it—contrary to the explicit assumption of uniqueness in the hypothesis of the theorem.

Secondly, no tangent hyperplane of a contour hypersurface, V_{n-1},[6] interior to **A** can cross the surface. For if it did, ϕ could be maximized (minimized) subject to the constraint only if it assumed a greater(lesser) value on the particular V_{n-1} than on any in its neighborhood. Then, as before, ϕ could be maximized(minimized) at any point of V_{n-1} subject to an infinitude of linear conditions, and the property of this paragraph is established.

Thirdly. Term the hyperplanes through a point tangent to contour hypersurfaces anywhere in **A** a fractional bundle. Then, the hyperplanes of a fractional bundle, other than the tangent, cross and are tangent to contour hypersurfaces on one side of the V_{n-1} through the point. To see this we pass through steps in some respects like, in others unlike, the reasoning of the two dimensional case. Since its tangent hyperplanes nowhere cross a V_{n-1}, the plane at any point touches on one side and the positive direction of the normal is unambiguously defined. Let ω be the angle, measured counterclockwisely[9], between the normal and a fixed direction, say that of a positive coordinate axis. Then, on the supposition that the hyperplanes of a bundle can be tangent in **A** on both sides of the V_{n-1} through the point, $\sin \omega$ will be positive for points on one side and negative on the other. For if, under these circumstances, $\sin \omega$ were positive or negative on both sides, at least one of the hyperplanes through the point under consideration would intersect the V_{n-1} which it touches. At each point of the intersection, ϕ could be maximized(minimized) subject to at least two constraints—the intersecting hyperplane and the tangent hyperplane to the contour hypersurface through the point—contrary to hypothesis. Or, if the intersecting and tangent hyperplanes coincide, it would be impossible to maximize(minimize) ϕ properly subject to *any* linear constraint—again contrary to hypothesis. Accordingly, $\sin \omega$ is positive on one side and negative on the other, and, in virtue of the assumed continuity of the

[5] A hyperplane will be understood to be of $n - 1$ dimensions.

[6] V_i will symbolize any hypersurface of intrinsic dimensionality i.

[9] See [9] on p. 106.

first derivatives of ϕ,[7] $\sin \omega$ vanishes throughout some continuum C of at least $n - 1$ dimensions. Everywhere in C, normals to contour hypersurfaces are parallel to the fixed direction.

If C happens to have n dimensions, a contradiction is quickly arrived at. For, then, contour hypersurface parts situate within C are hyperplanes since their normals are everywhere parallel to a fixed direction. At each point of such a hyperplane, ϕ cannot be properly maximized (minimized) subject to *any* linear constraint—contrary to hypothesis.

Should C have exactly $n - 1$ dimensions, the intersection of a contour hypersurface V_{n-1} and C is a V_{n-2} in n dimensional space. Also, the $n - 2$ dimensional tangent flat spaces of V_{n-2}, since they are embedded in tangent hyperplanes of V_{n-1}, are everywhere perpendicular to the fixed direction. The intersection of V_{n-2} and a hyperplane having the point Q in common with V_{n-2}, passing through the fixed direction, and failing to contain V_{n-2}[8] is a V_{n-3} in $n - 1$ dimensional space. And the $n - 3$ dimensional tangent flat spaces of V_{n-3} are all perpendicular to the fixed direction. So that the figure is reduced to a similar one in $n - 1$ dimensions. After $n - 3$ reductions, we arrive at a V_1, in three dimensional space, whose tangents are everywhere perpendicular to the fixed direction. Consequently, the V_1 is a plane curve lying in a plane normal to the fixed direction. And, since the tangent hyperplane to V_{n-1} at Q is also perpendicular to this direction, it contains V_1. Whence, since V_1 is a part of V_{n-1}, it is impossible to *properly* maximize (minimize) ϕ at Q subject to any linear constraint—contrary to hypothesis.

Fourthly, for some small swath ϕ assumes higher(lower) values on that side of the contour hypersurface through a point on which the hyperplanes of the fractional bundle, other than the tangent hyperplane, touch than on the opposite side. Otherwise, in this swath, ϕ would

[7] The use of $\sin \omega$, instead of a function involving the curvature properties of the contour hypersurfaces, shows that assumption of continuity on the parts of the second derivatives of ϕ is unnecessary. Their existence is all the theorem requires. The earlier proof for $n = 2$ can be modified to comply with this observation. Indeed, a revised demonstration is contained in the proof for the general n being given.

[8] In case the hyperplane contains V_{n-2}, we can terminate the argument immediately. For, then, the $n - 2$ dimensional tangent flat spaces of V_{n-2} are everywhere perpendicular to the fixed direction in $n - 1$ dimensional space, and V_{n-2} is itself a flat space. Whereupon slight reflection convinces one that ϕ cannot be properly maximized(minimized) subject to any linear constraint at any point of V_{n-2}.

assume higher(lower) values along the contour hypersurfaces which the tangent hyperplane cuts than along the hypersurface which it touches. And ϕ would be minimized(maximized) rather than maximized(minimized) subject to the linear constraint represented by the tangent hyperplane at the point under consideration.

Now to the theorem proper. Each point in the portion of the hyperplane tangent to the contour hypersurface at P contained in **B** is in one-to-one correspondence with a fraction of the bundle of hyperplanes through Q in **A**. Of these hyperplanes only the one associated with P, and tangent to the contour hypersurface through Q, does not cross the surface. The others do and are for some small swath tangent to contour hypersurfaces on one side of the surface through Q and with higher(lower) values of ϕ. And so the values of ψ at the points of some $n - 1$ dimensional region about P in the initial hyperplane in **B**, which are equal to these values of ϕ, are higher(lower) than the value of ψ at P. Therefore, ψ is minimized(maximized) at P subject to the linear constraint represented by the initial hyperplane.

The geometric proof just adduced establishes that ψ satisfies the second order, as well as the first order, conditions for a minimum (maximum).

ANALYTIC DEMONSTRATION THAT THE FIRST ORDER CONDITIONS FOR THE EXTREMIZATION OF ϕ SUBJECT TO THE ORIGINAL CONSTRAINT AND OF ψ SUBJECT TO THE INVERSE CONSTRAINT ARE IDENTICAL

We employ the umbral or dummy symbol notation throughout.

We are given $\phi_i = \lambda p_i$ $(i = 1, \cdots, n)$ and are to prove $\psi_j = \sigma q_j$ $(j = 1, \cdots, n)$ where λ and σ are Lagrange multipliers.

It is easy to establish that

$$\phi_i q_r p_{rj} = \phi_j q_s p_{si} \qquad \begin{matrix}(r \text{ and } s \text{ umbral}) \\ (i, j = 1, \cdots, n)\end{matrix} \tag{1}$$

is an identity satisfied for all ϕ functions possessing second partial derivatives. For

$$p_s = \lambda^{-1}\phi_s \qquad \text{So that,} \tag{2}$$

$$p_{si} = \lambda^{-2}\left(\lambda\phi_{si} - \frac{\partial\lambda}{\partial q_i}\phi_s\right) \qquad \text{But,} \tag{3}$$

$$\lambda = m^{-1}\phi_t q_t \qquad (t \text{ umbral}). \text{ And} \tag{4}$$

$$\frac{\partial\lambda}{\partial q_i} = m^{-1}(\phi_i + \phi_{ti} q_t) \qquad \text{Hence,} \tag{5}$$

$$p_{si} = m^{-1}\lambda^{-2}\{q_t\phi_t\phi_{si} - \phi_i\phi_s - q_t\phi_s\phi_{ti}\} \tag{6}$$

So that the right side of (1) becomes

$$\phi_i q_s p_{si} = -m^{-1}\lambda^{-2}\phi_s\phi_j q_s\phi_s , \tag{7}$$

an expression symmetrical in i and j.

Now if $\psi = \phi[q_1(p_1, \cdots, p_n), \cdots, q_n(p_1, \cdots, p_n)]$, then $\phi = \psi[p_1(q_1, \cdots, q_n), \cdots, p_n(q_1, \cdots, q_n)]$ and

$$\phi_i = \psi_s p_{si} \qquad (s \text{ umbral}). \tag{8}$$

But from (1),

$$\phi_i = \sigma q_s p_{si}, \tag{9}$$

where σ is a factor of proportionality. Hence, subtracting (9) from (8), we have the equations,

$$(\psi_s - \sigma q_s)p_{si} = 0 \qquad \begin{matrix}(s \text{ umbral}) \\ (i = 1, \cdots, n).\end{matrix} \tag{10}$$

But determinant $(p_{si}) = \dfrac{\partial(p_1, p_2, \cdots, p_n)}{\partial(q_1, q_2, \cdots, q_n)}$ cannot vanish or solution for the q's in terms of the p's would be impossible and our entire labor meaningless. Hence,

$$\psi_s = \sigma q_s \qquad (s = 1, \cdots, n), \tag{11}$$

as was to be shown.

Since $p_s q_s = m$ must always obtain, it follows that $\sigma = m^{-1}\psi_s p_s = m^{-1}\sum_{s=1}^{n}\psi_s p_s$. This completes the demonstration.

APPLICATION TO DIFFERENTIAL EQUATIONS

As is well known the total (Pfaffian) differential equation

$$\sum_{i=1}^{n} P_i dq_i = 0 \tag{1}$$

possesses an integral when $\frac{1}{2}(n-1)(n-2)$ independent conditions of the form,

$$P_i\left(\frac{\partial P_j}{\partial q_k} - \frac{\partial P_k}{\partial q_j}\right) + P_j\left(\frac{\partial P_k}{\partial q_i} - \frac{\partial P_i}{\partial q_k}\right) + P_k\left(\frac{\partial P_i}{\partial q_j} - \frac{\partial P_j}{\partial q_i}\right) = 0, \tag{2}$$

are satisfied. But often the formal integration is difficult. In such

cases if we put

$$p_i = \frac{mP_i}{\sum_{s=1}^{n} q_s P_s}, \tag{3}$$

where m is a constant not equal to zero and solve for the q's in terms of the p's obtaining

$$q_j = Q_j(p_1, \cdots, p_n) \qquad (j = 1, \cdots, n), \tag{4}$$

it may happen that the equation

$$\sum_{j=1}^{n} Q_j dp_j = 0 \tag{5}$$

is more easily integrable. The duality just established between ϕ and ψ assures us that the coefficients of (5) satisfy the integrability conditions and that if a solution is obtained in the form

$$\Psi(p_1, \cdots, p_n) = 0, \tag{6}$$

a solution of (1) can be had by substituting the values of the p's in terms of the q's as given by (3).

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[9] The angle between a directed line through the coordinate origin parallel to the positive direction of the normal and the positive coordinate axis can be taken. Counterclockwise rotation in the plane of this line and the coordinate axis is defined as positive rotation relative to a specified direction along a second coordinate axis.

THE VISUALIZATION OF DOMAINS OF THE THEORY OF FUNCTIONS OF TWO COMPLEX VARIABLES

By Stefan Bergman

1. The interpretation of a complex number as a point of the plane enables us to formulate geometrically many analytic relations in the theory of functions of one complex variable. It appears, therefore, that this fact is of fundamental importance in this theory. When employing this geometric representation we can do it either by defining geometric figures by equations or inequalities and interpreting the analytic relations among the variables as geometric properties of the figures; or by drawing these figures and operating with them directly. For example, we can introduce a circle* by its equation $x^2 + y^2 = 1$ and discuss its properties from this equation; or we can draw it and work with this figure.

We can often easily deduce properties of manifolds from a graphic representation and in the theory of functions of one complex variable this not only offers a convenient and simple way of explaining certain theorems, but further stimulates the investigator to introduce important new notions which lead to a fruitful association of ideas.

We shall refer to the ability to operate with geometric figures as "geometric intuition."

While in the last century the theory of one variable reached a high degree of development and found fruitful applications in different branches of mathematics, the progress of the generalization of this theory to the case of two variables was much slower even though in various branches of mathematics questions of this field arose and many mathematicians have dealt with various problems of this theory.

One of the difficulties that hinders the progress of this field emerges from the fact, that the analogous geometrical interpretation leads to four-dimensional figures. As we associate to a complex number a POINT

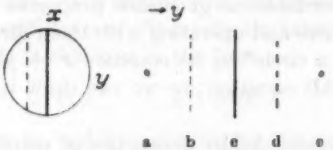
* The manifold of points of the plane, which are equally distant from the point $\{x = 0, y = 0\}$. Obviously a drawn figure is as a rule only an approximation of geometrical objects; since, for instance, any drawn curve must possess a certain thickness. In order to deduce certain geometrical properties from the drawn figures we must draw on our imagination. We do not, however, wish to deal with this question in this paper.

OF THE PLANE it is necessary to associate to a pair $\{\zeta_1, \zeta_2\}$ of complex numbers A POINT OF THE FOURDIMENSIONAL SPACE.

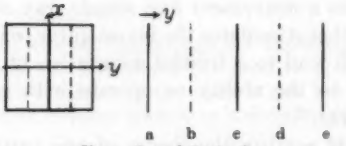
Since we cannot easily visualize objects in a fourdimensional space and do not have the ability to operate easily with figures of this space, it becomes necessary to create in an artificial way the missing "geometrical intuition" so as to be able to construct models of these figures.

We should naturally require of them that they render to a certain degree the same services as the pictures in the case of one variable. In the next paragraph we shall describe models of this nature.

2. The method used for giving us a visualization of fourdimensional space consists in interpreting the fourth dimension as the time coordinate. Therefore we receive threedimensional moving figures for the



FIGS. 1 and 1a to 1e



FIGS. 2 and 2a to 2e

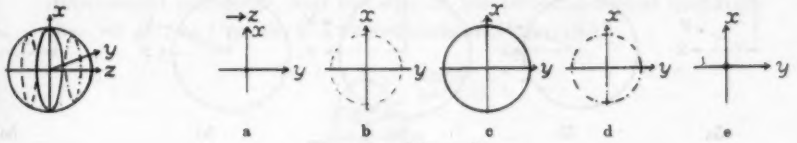
above mentioned models. We proceed to explain our method by an analogy:

Suppose a being, capable of imagining only onedimensional figures, who desires to construct in an analogous way models of twodimensional figures, for example the circle $x^2 + y^2 < 1$, or the square $|x| < 1$, $|y| < 1$. Let us assume that a film be made with each still representing a time intersection of the object in question. In the case of the circle the following film will appear: at first a point appears on the screen; it then transforms itself into a straight line. The length of this line grows until the line is equal in length to double the radius of the circle. From then on the length of the line diminishes, until it becomes a point, and finally disappears completely. In the case of a square there suddenly appears on the screen a line which remains fixed for some-

time and then just as suddenly disappears. In illustrations No. 1a–1e and No. 2a–2e "stills"* are shown, of our supposed film.

Let us suppose now, that a being, who can imagine twodimensional figures only, desires to construct the models of threedimensional figures. In the case of a sphere $x^2 + y^2 + z^2 < 1$ a point will appear at first on the screen; it then transforms itself into a circle, the radius of the circle increases until the radius reaches its maximum value (= 1). The radius then begins to diminish until the circle becomes a point and then vanishes completely. (See figures No. 3a–3e.)

In the case of the cylinder ($x^2 + z^2 < 1, |y| < 1$) we see at first a straight line, which transforms itself into a rectangle. Its breadth remains fixed but its height increases a certain maximum value. The



FIGS. 3 and 3a to 3e

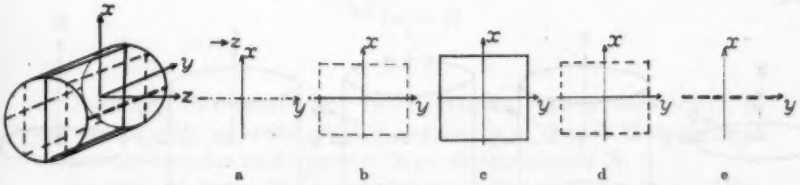


FIG. 4

altitude of the rectangle then begins to diminish until we see a straight line again, and finally vanishes. Naturally, for those who can imagine two and threedimensional figures, it is easier to operate with figures No. 1, 2, 3, 4 than with our models No. 1a–1e, 2a–2e, 3a–3e, 4a–4e; but with sufficient practice† it is possible to see different properties of two- and threedimensional figures with the aid of our models.

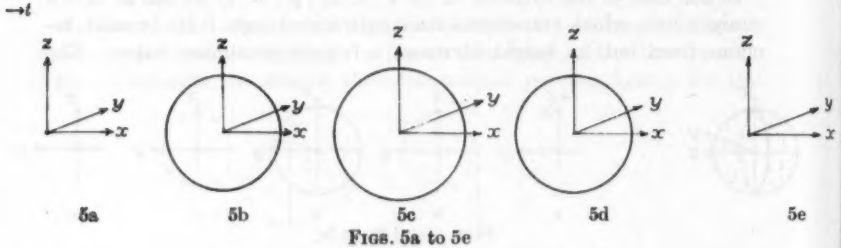
In the case of fourdimensional figures, for example, the hypersphere ($x^2 + y^2 + z^2 + t^2 < 1$) or the bicylinder ($x^2 + y^2 < 1, z^2 + t^2 < 1$), which we cannot easily imagine, we can construct analogously film-models. (As the designs on the screen are twodimensional, it is natu-

* The intersection of a domain $\mathfrak{B}$ with the space $t = t_0 = \text{const.}$, i.e. the picture which we see on the screen at the moment t_0, we shall call the "still of $\mathfrak{B}$ for t_0."

† Such as learning a foreign language which we wish to speak fluently.

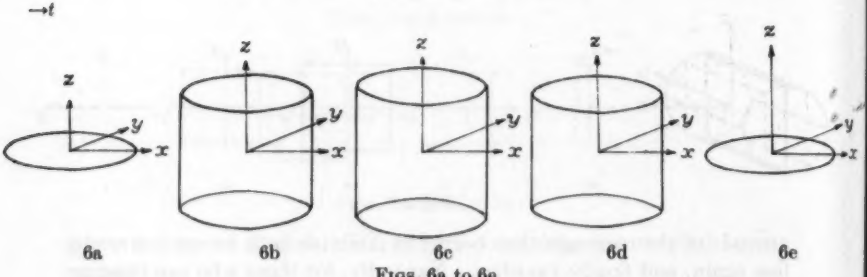
rally necessary that the spectator visualizes that these twodimensional designs represent threedimensional bodies.) In the case of a hypersphere we shall see at first a point, which transforms itself into a sphere. The radius of the sphere increases, takes its maximum value, than diminishes until it becomes a point again and then vanishes. (See figures 5a–5e.)

In the case of a bicylinder, a circular-disk at first appears; it transforms itself into a cylinder with a circular base and an altitude of which grows until it reaches a maximum value. The altitude of the cylinder



5a 5b 5c 5d 5e

FIGS. 5a to 5e



6a 6b 6c 6d 6e

FIGS. 6a to 6e

then diminishes, it transforms itself into a circular-disk, and then vanishes. (See figures 6a–6e.)*

3. In order to give a model of other fourdimensional bodies and to illustrate how to work with these models, we shall explain in the above mentioned way, the notion of a "DOMAIN WITH DISTINGUISHED SURFACE" and prove one of the properties of these domains.

The importance of the "principle of maximum" in the theory of func-

* Illustrations of other figures are given in: "Zur Veranschaulichung der Kreiskörper und Bereiche mit ausgezeichneter Randfläche" Jahresbericht deutsch. Math.-Verein, **42**, 1933, p. 238–252.

tions of one complex variable is very well known. The "principle of maximum" is: an analytic function of one complex variable takes its maximum modulus on the boundary of the considered domain. As mentioned, above, a function of two complex variables must be defined in a fourdimensional domain. The boundary of such a domain is a threedimensional manifold. We can "see" this last fact, for example, in the following manner:

The boundary of a domain (as we can verify in figures No. 1a–1e, 2a–2e, 3a–3e, 4a–4e) consists of the sum of the boundaries of the stills for every t, and of the two domains that appear for the maximum and minimum values of t. In the case of a fourdimensional domain the stills for $t = \text{const.}$ are threedimensional domains, their boundaries twodimensional manifolds, and the sum of the twodimensional manifolds that we get, as t varies, is a threedimensional manifold.

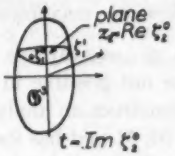


FIG. 7

PRINCIPLE OF MAXIMUM FOR TWO VARIABLES.—*Every function* $f(\zeta_1, \zeta_2)$ *of two complex variables which is regular in a domain* $\mathfrak{B}$ *assumes its maximum-modulus with respect to* $\mathfrak{B}$ *on the boundary of* $\mathfrak{B}$.

In order to prove this we shall assume that it is not true. Then, there exists an interior point $\{\zeta_1^0, \zeta_2^0\}$ of $\mathfrak{B}$, such that $|f(\zeta_1^0, \zeta_2^0)|$ is greater than $|f(\zeta_1, \zeta_2)|$ at some other points of $\mathfrak{B}$.

Let $\mathfrak{g}^{3*}$ (see figure No. 7) represent the still of $\mathfrak{B}$ for $t = \mathbf{Im}\zeta_2^0$ (Im = Imaginary part). $\{\zeta_1^0, \mathbf{Re}\zeta_2^0\}$ is an interior point of the intersection of $\mathfrak{g}^3$ and of the plane $t = \mathbf{Re}\zeta_2^0$. But this last intersection is a two dimensional domain and $f(\zeta_1, \zeta_2)$, for $\zeta_2 = \zeta_2^0$, is an analytic function of one compex variable. Therefore, by the principle of maximum for functions of one complex variable, $|f(\zeta_1, \zeta_2^0)|$ must take its maximum at a point on the boundary of the indicated twodimensional domain; for instance, at the point $\{\zeta_1^1\}$. We have $|f(\zeta_1^1, \zeta_2^0)| \geqq |f(\zeta_1^0, \zeta_2^0)|$. But point $\{\zeta_1^1, \zeta_2^0\}$ belongs to the boundary of $\mathfrak{g}^3$, and therefore also to the boundary of $\mathfrak{B}$. $|f|$ therefore takes its maximum at a point on the boundary, which contradicts our hypothesis.

* The superior index n, $0 < n < 4$, indicate the dimension of a point manifold.

In many other cases, however, we see that this generalisation of the principle of maximum does not lead us to analogous consequences, as in the case of one variable. This is connected with the fact that the threedimensional boundary in the case of two variables possesses properties essentially different from the boundary line in the case of one variable.

The lemma of Schwarz, which is the basis of many investigations in the case of one variable, tells us that if $f(\zeta)$ is regular in the unit circle $\mathfrak{C}^2$ vanishes at the point $\zeta = 0$, and if $|f(\zeta)| \leqq 1$ in $\mathfrak{C}^2$ then $|f(\zeta)| \leqq |\zeta|$ in $\mathfrak{C}^2$. The proof of this lemma is based on the fact that there exists a function $w(\zeta, 0) \equiv \zeta$, regular in $\mathfrak{C}^2$, which vanishes at and only at the point $\zeta = 0$ and whose modulus is equal to 1 on the whole boundary of $\mathfrak{C}^2$. $\dfrac{f(\zeta)}{w(\zeta, 0)}$ is regular in the circle and in view of the principle of maximum $\left|\dfrac{f(\zeta)}{w(\zeta, 0)}\right|$ takes its maximum $(=1)$ on the boundary. Therefore we have: $|f(\zeta)| \leqq |w(\zeta, 0)| = |\zeta|$.

But such conclusions are not possible in the case of two variables, since it is not possible to construct an analytic function $w(\zeta_1, \zeta_2; 0, 0)$ that vanishes at the point $\{0, 0\}$ and has its modulus equal to 1 on the boundary of the domain. (In regard to the mentioned theorem, an analytic function of two variables must take on every value on the boundary that it takes on *in* the domain. Therefore $w(\zeta_2, \zeta_2; 0, 0)$ vanishes at least at one point on the boundary and at this point $|w(\zeta_1, \zeta_2; 0, 0)| \neq 1$).

In order to obtain efficacious applications of the generalization of the principle of maximum, it is necessary to generalize it in another manner: for instance, to prove that $|f|$ takes its maximum on a certain part of the boundary, and that this part can be so chosen that certain functions exist that vanish in the domain without vanishing on the indicated part of the boundary.

If there exists a REAL part of the border $\mathfrak{F}^n$, $n = 2$ or 3 such that for every function f regular in the domain $|f|$ takes its maximum not only on the boundary, but more specifically on $\mathfrak{F}^n$, we call it the "distinguished boundary manifold"* of the domain.

Such a manifold cannot exist for an arbitrary domain, as is obvious in the case of the hypersphere where by symmetry, no point of the boundary can be distinguished from any other. We shall show, however, that there exists a special class of domains, possessing an $\mathfrak{F}^2$.

* In the case $n = 2$ we shall call $\mathfrak{F}^2$ a "distinguished boundary surface" or more briefly a "distinguished surface."

4. We shall go on to a detailed description of these domains, and to do this we shall introduce a new conception.

We shall call an "ANALYTIC SURFACE" ($\mathfrak{S}^2$), a twodimensional point manifold, one which can be represented in the form

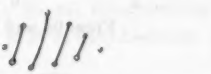
$$\zeta_k = h_k(Z),\ k = 1, 2 \tag{4.1}$$

where Z runs through the domain of definition of h_k and $h_k(Z)$, $k = 1, 2$ are analytic functions of one complex variable.

If Z runs through only a domain, $\mathfrak{B}^2$, of the complex plane then $\{h_1(Z), h_2(Z)\}$ runs through a part of the analytical surface $\mathfrak{S}^2$, which we shall call "A SEGMENT OF AN ANALYTIC SURFACE" and which we shall denote in the future by $\mathfrak{Z}^2$.

To every point Z of $\mathfrak{B}^2$ corresponds a point $\{h_1(Z), h_2(Z)\}$ of $\mathfrak{Z}^2$ and vice versa.

We shall call the set of points of $\mathfrak{S}^2$ that corresponds to the boundary of $\mathfrak{B}^2$, the "BOUNDARY" of $\mathfrak{Z}^2$ and shall denote it by $\mathfrak{i}^1$. The functions $f(\zeta_1, \zeta_2)$ and $F(Z) = f[h_1(Z), h_2(Z)]$ take on the same values in corresponding points. As $F(Z)$ takes its maximum modulus relative to $\mathfrak{B}^2$ in a point of its boundary $\mathfrak{b}^1$, $f(\zeta_1, \zeta_2)$ takes its maximum relative to $\mathfrak{Z}^2$ in a point of $\mathfrak{i}^1$.



a b c d e f

FIGS. 8a to 8f

A segment of an analytic surface will appear in our representation as a moving line curve so that for every t of an interval we see on the screen a line-curve. (See figures No. 8a–8f). The sum of the endpoints of these lines forms the curve $\mathfrak{i}^1$.

We call an "ANAYLTIC HYPERSURFACE" the sum of analytical surfaces, which depend continuously upon a real parameter. An analytic hypersurface will be represented by

$$\zeta_k = h_k(Z, \lambda),\ 0 \leqq \lambda \leqq 2\pi,\ |Z| < \infty \tag{4.2}$$

where $h_k(Z, \lambda)$ are continuous functions of Z and λ, and represent for every λ an analytic function of Z.

We shall call a "LAMINA" the particular surface, which we get when we give a fixed value to λ. The still of an analytic hypersurface is (generally) a surface, which extends to infinity to both directions (see

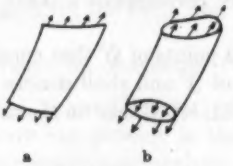
figs. 9a and 9b, where arrows point out, that the surface extends to infinity in the indicated direction).

A sum $\overset{2\pi}{\underset{\lambda=0}{S}} \mathfrak{F}^2(\lambda)$ of segments of analytic surfaces is called a "SEGMENT OF AN ANALYTIC HYPERSURFACE" and will be designated by $\mathfrak{f}^3$.

A segment of an analytic hypersurface can be written in the form

$$\zeta_k = h_k(Z, \lambda), \; 0 \leqq \lambda \leqq 2\pi, \; Z \in \mathfrak{B}^2(\lambda) \tag{4.3}$$

where $\mathfrak{B}^2(\lambda)$ means a domain of the complex Z-plane which varies with λ. Further $h_k(Z, \lambda)$ is, as mentioned above, a continuous function of Z and λ, which is an analytic function of Z for any fixed λ and such, that to each point of $\mathfrak{F}^2(\lambda)$ corresponds one and only one point of $\mathfrak{B}^2(\lambda)$.



FIGS. 9a and 9b

FIG. 10

We call the sum $\overset{2\pi}{\underset{\lambda=0}{S}} \mathfrak{i}^1(\lambda)$ of the images $\mathfrak{i}^1(\lambda)$ of $\mathfrak{b}^1(\lambda)$ the MANTLE SURFACE OF THE SEGMENT OF ANALYTIC HYPERSURFACE. $\mathfrak{b}^1(\lambda)$ denotes the boundary of $\mathfrak{B}^2(\lambda)$, In figures No. 9a and 9b the stills of the mantle surface are designated by arrows.

If we cut away a portion of an analytical hypersurface, its boundary consists (except in some degenerate cases) of a finite number of lamina and of the mantle surface.

This is obvious if we study figures 9a and 9b.

If $f(\zeta_1, \zeta_2)$ is regular at every point of a segment of an analytic hypersurface, $|f|$ takes its maximum with respect to $\mathfrak{f}^3$ in a point of the mantle surface of $\mathfrak{f}^3$.

Proof. $|f|$ must take its maximum in any surface $\mathfrak{F}^2(\lambda)$ of $\mathfrak{f}^3$ for

instance in $\mathfrak{Z}^2(\lambda^0)$. On the other hand from the result mentioned above $|f|$ takes its maximum with respect to $\mathfrak{Z}^2(\lambda^0)$ on the border $\mathfrak{i}^1(\lambda^0)$ of $\mathfrak{T}^2(\lambda^0)$. According to our definition $\mathfrak{i}^1(\lambda^0)$ belongs to the mantle surface of $\mathfrak{i}^3$.

5. Let us be given a finite number of analytic hypersurfaces which divide a part of the fourdimensional space into cells in such a manner, that there exists at least one cell denoted by $\mathfrak{M}$ lying entirely in a finite part of the space. Figure No. 10 represents a still of such cell. (We have drawn stills of those parts $\mathfrak{i}^3$ of $\mathfrak{h}^3$, which belong to the boundary of the stills of $\mathfrak{M}$).

Every segment $\mathfrak{i}^3_\nu$, of the analytic hypersurface $\mathfrak{h}^3_\nu$ which belongs to the boundary of $\mathfrak{M}$ is bounded by the intersection of $\mathfrak{h}^3_\nu$ with other hypersurfaces $\mathfrak{h}^3_k$, $k = 1, 2, \cdots \nu - 1, \nu + 1, \cdots n$. The boundary of $\mathfrak{i}^3_\nu$ consists, as mentioned above, of the end laminas and of its mantle surface.

We shall designate the mantle surface of $\mathfrak{i}^3_\nu$ by $\mathfrak{G}^2_\nu$. In the figure No 10 the still of $\mathfrak{G}^2_\nu$, is the curve, which bounds the still of $\mathfrak{i}^3_\nu$. (We notice, that any point of $\mathfrak{G}^2_\nu$, as an intersection of $\mathfrak{h}^3_\nu$ with another hypersurface, for instance, $\mathfrak{h}^3_k$, $k \neq \nu$, also belongs to the mantle surface of the corresponding $\mathfrak{i}^3_k$).

$\mathfrak{M}$ is a domain with a distinguished surface, namely $\mathfrak{F}^2$ sum of all $\mathfrak{G}^2_\nu$, $\nu = 1, 2, \cdots n$.*

It is a fact that every function f, regular in $\mathfrak{M}$, must take its maximum-modulus on the boundary, and, since the boundary of $\mathfrak{M}$ consists of a finite number of segments $\mathfrak{i}^3_k$, $k = 1, 2, \cdots n$ of analytic hypersurfaces the maximum-modulus must be taken on one of $\mathfrak{i}^3_1$, $\mathfrak{i}^3_2$, $\cdots$ $\mathfrak{i}^3_n$, for instance, on $\mathfrak{i}^3_\nu$. On the other hand $|f|$ attains its maximum with respect to $\mathfrak{i}^3_\nu$, in the $\mathfrak{G}^2_\nu$, which belongs to the distinguished surface $\mathfrak{F}^2$ of $\mathfrak{M}$.

6. It is now easy to show, that the introduction of domains with a distinguished surface permits us to give a generalization of the Lemma of Schwarz. It is evidently possible to construct functions which vanish in $\mathfrak{M}$, without vanishing on the distinguished surface $\mathfrak{F}^2$. For instance in figure No. 10 we see a still of a domain $\mathfrak{M}$ (represented as described

* I wish to mention that Mr. Hilsenrath has recently constructed a number of threedimensional bodies forming the stills of a domain with distinguished boundary surface. These models had proved to be of great value from the pedagogical point of view as well as for purposes of investigation.

before) with the distinguished surface $\mathfrak{F}^2$. The surface $\mathfrak{F}^2$ appears as a curve which is formed by the intersections of the stills of $\mathfrak{i}^2_\zeta$. The still of the analytic surface $w(\zeta_1, \zeta_2) \equiv \zeta_1 \equiv 0$ will appear as the line $x = 0, y = 0$.

We see that the function ζ_1 does not vanish on $\mathfrak{F}^2$. Therefore if we consider a function $f(\zeta_1, \zeta_2)$, regular in $\mathfrak{M}$, such that $\frac{f(\zeta_1, \zeta_2)}{\zeta_1}$ is also regular in $\mathfrak{M}$, we obtain by the principle of the maximum in a domain with a distinguished surface (every function takes its maximum modulus on the distinguished surface)

$$\left|\frac{f(\zeta_1, \zeta_2)}{\zeta_1}\right| \leqq \operatorname*{Max}_{\{\zeta_1,\zeta_2\} \in \mathfrak{F}^2} \left|\frac{f(\zeta_1, \zeta_2)}{\zeta_1}\right| \leqq \frac{\operatorname*{Max}_{\{\zeta_1,\zeta_2\} \in \mathfrak{F}^2} |f(\zeta_1, \zeta_2)|}{\operatorname*{Min}_{\{\zeta_1,\zeta_2\} \in \mathfrak{F}^2} |\zeta_1|} \tag{6.1}$$

As the min of $|\zeta_1|$ on $\mathfrak{F}^2$ is positive, the expression given on the right of (6.1) is finite and we therefore have in $\mathfrak{M}$

$$|f(\zeta_1, \zeta_2)| \leqq \mu |\zeta_1|, \qquad \mu = \frac{\operatorname*{Max}_{\{\zeta_1,\zeta_2\} \in \mathfrak{F}^2} |f(\zeta_1, \zeta_2)|}{\operatorname*{Min}_{\{\zeta_1,\zeta_2\} \in \mathfrak{F}^2} |\zeta_1|} \tag{6.2}$$

If we consider an arbitrary domain $\mathfrak{B}$ (without a distinguished surface) we must take the minimum of $|\zeta_1|$ on the whole threedimensional boundary of $\mathfrak{B}$ instead of the minimum on $\mathfrak{F}^2$, and this minimum is equal to 0. We can not obtain a bound for $|f|$ in $\mathfrak{B}$ analogous to (6.2).

7. It is well known that there exists in the theory of functions of one complex variable other theorems involving the boundary curve (the integral formula of Cauchy, the theorem of Rouché and others). These theorems, for instance lemma of Schwarz are the starting points of different theories (for instance the theory of meromorphic functions).

We have introduced the notion of the distinguished surface as the generalization of the boundary curve. The question therefore arises whether it is possible to find analogues of the above mentioned theorems in the case of two complex variables and whether it is possible to develop a theory of analytic and meromorphic functions of two complex variables from these results.

If one tries to use a technique analogous to that of the theory of functions of one complex variable one observes immediately that the class of functions of two complex variables is in a certain sense too small to give a generalization of the different formulas which are so useful in the theory of functions of one complex variable. E.g. the existence of

the Green's function, one of fundamental tools in the theory of function of one complex variable, is based on the fact that it is possible to find for every continuous function given on the boundary curve a harmonic function which assumes preassigned values on the boundary. On the other hand one can easily define a continuous function on the distinguished surface such that there *does not* exist a biharmonic* function which takes on these boundary values on the distinguished boundary surface. It results from this that in addition to the biharmonic function it is useful to consider a certain class of functions, called *extended class of functions* which possess certain properties necessary for our purposes. This class of functions depends on the domain. E.g. in the case of the bicylinder the double harmonic functions (functions satisfying equations

$$\frac{\partial^2 u}{\partial \xi_k^2} + \frac{\partial^2 u}{\partial \eta_k^2} = 0, \qquad k = 1, 2) \tag{7.1}$$

forms the extended class of functions. The biharmonic functions satisfy in addition to (7.1) the equations

$$\frac{\partial^2 u}{\partial \xi_1 \partial \xi_2} + \frac{\partial^2 u}{\partial \eta_1 \partial \eta_2} = 0, \qquad \frac{\partial^2 u}{\partial \xi_1 \partial \eta_2} - \frac{\partial^2 u}{\partial \xi_2 \partial \eta_1} = 0. \tag{7.2}$$

The introduction of the functions of the extended class enables us to give formulas analogous to those in the theory of functions of one complex variable, namely the generalized Green's function, a formula analogous to the Cauchy's formula† etc.

The question arises whether in spite of the changed situation it is possible now to repeat the technique used in the case of one variable studying the analytic and meromorphic functions of two complex variables. As a matter of fact this is possible to a certain degree. The investigation of these questions forms the contents of the theory of distinguished surfaces.‡

I wish to express my thanks to Dr. A. Gelbart for his assistance in the writing of the present paper.

CAMBRIDGE, MASS.

* We understand by biharmonic function the real or imaginary part of an analytic function of two complex variables.

† See "Über eine in gewissen Bereichen mit Maximumflache gültige Darstellung . . ." Math. Zeitsch., **39**, 1934, p. 76–94 and 605–608; "Über eine Integraldarstellung, der Funktionen zweier komplexen Veränderlichen," Recueil mathém., **1** (**43**), 1936, p. 851–862.

‡ See "Über meromorphe Funktionen von zwei komplexen Veränderlichen," Compositio mathematica, **6**, 1939, p. 305–335.

ON A THEOREM OF RENGELS

D. C. SPENCER

1. We denote by $\phi(r)$ the annulus $0 \leq r \leq |z| \leq 1$ slit along concentric circular arcs with $z = 0$ as center. If $r = 0$, we write simply ϕ.

Suppose that $w = f(z)$ maps ϕ on a Riemann domain with $f(0) = 0$, $|f'(0)| = 1$; and let D be the maximum distance of the boundary of W from $w = 0$. Rengels[1] has shown that if W is schlicht, then $D > 1$, unless f is a rotation; and the same theorem has lately been rediscovered by Schiffer[2]. My purpose here is to generalize this theorem.

We denote by $\sigma_1, \sigma_2, \cdots$ the boundary elements of W corresponding to the concentric circular slits of ϕ, by B the remainder of the boundary (which corresponds to $|z| = 1$). Let U be the domain bounded by B, and let U^* be the portion of U which can be seen from $w = f(0)$, B^* the boundary of U^*; and let W^* be the sub-domain of W every point of which can be seen either from $f(0)$, or from a point of B^* looking toward $f(0)$, if any one σ_i is opaque.

We say that a Riemann domain G is mean 1-valent (with respect to the origin) if the area (multiply covered points being counted multiply) of that part of G lying in *any* circle with center 0 does not exceed the area of the circle. I prove the following result:

THEOREM 1. *If W^* is mean 1-valent, then $D > 1$ unless $f = e^{i\alpha}z$.*

When there are no slits and W is simply-connected, the theorem has been proved by Bermant[3]. In this case W^* is schlicht[4], and no valency hypothesis is necessary.

[1] E. Rengels, "Über einige Schlitztheoreme der korformen Abbildung," *Schr. math. Sem. Univ. Berlin*, 1, 4(1933), 141–162.

[2] M. Schiffer, "Sur un theoreme de la reprèsentation conforme, *C. R. Acad. Sci., Paris*, 207(1938), 520–522. Schiffer states the theorem in the following form: Let G be a bounded domain containing $z = 0$, and let F be the family of functions regular and schlicht in G, with $f(0) = 0$, $|f'(0)| = 1$. Let $M(f)$ be the maximum modulus of f in G, and let $m = \operatorname{Min}_{f \in F} M(f)$. Then $M(f) > m$ unless f maps G on the circle $|z| < m$ slit along concentric circular arcs. Schiffer proves the theorem by a variational method.

[3] A. Bermant, "Sur quelques propriétés des fonctions régulières," *C. R. Acad. Sci. U.R.S.S.*, 18(1938), 137–140; "Remarque sur le lemme de Schwarz," *C. R. Acad. Sci., Paris*, 207(1938), 31–33.

[4] Suppose only that W^* is schlicht (but possibly multiply connected), and let V_1^* be the area of W^*, V_2^* the area of the transform of W^* by f^{-1}. Then it can be shown that $V_1^* V_2^* > \pi^2 |f'(0)|^2$, unless $f = az$. This result, however, is more

2. We write $\zeta = \xi + i\eta$. Let G_1 and G_2 be two Riemann domains such that $g(\zeta)$ maps G_1 on G_2; and let $q(\eta)$ be a set of points of G_1, defined for each η of a set H, which lie over the line $\eta =$ constant. We write

$$E_1 = \sum_{\eta \in H} q(\eta) \qquad L(\eta) = \int_{q(\eta)} |g'|\, d\xi,$$

$$V_1 = \int_{E_1} d\xi\, d\eta \qquad V_2 = \iint_{E_1} |g'|^2\, d\xi\, d\eta.$$

We suppose that $L(\eta) \geq \lambda$ for almost all $\eta \in H$. Then by the inequality of Schwarz we have

$$(\lambda m H)^2 \leq \left[\int_H L(\eta)\, d\eta\right]^2 = \left[\iint_{E_1} |g'|\, d\xi\, d\eta\right]^2$$

$$\leq \left[\iint_{E_1} |g'|^2\, d\xi\, d\eta\right] \cdot \left[\iint_{E_1} d\xi\, d\eta\right] = V_1 \cdot V_2, \tag{2.1}$$

and it is easy to see that equality can occur only if $V_1 = 0$, or $g(\zeta) = a_0 + a_1\zeta$.

Lemma.

$$V_1 \cdot V_2 \geq (\lambda m H)^2, \tag{2.2}$$

and there is equality only if $g(\zeta) = a_0 + a_1\zeta$.

The lemma is an expression in new form of an idea which was first systematically applied by Grötzsch[5], so far as I know. It was shortly afterward rediscovered and developed by Ahlfors[6], and similar ideas have been employed by Rengels (*loc. cit.*) and by other writers. The various forms in which the inequality has appeared are, however, more specialized than the one given here. For example, as developed by Rengels, one of the domains is a rectangle. Furthermore, the inequality has appeared in forms so different as to be scarcely recognizable (cf., e.g., Grötzsch, *loc. cit.*). At first glance it is rather remarkable that such an extremely simple idea should form the basis for so many interesting results. But of course (2.2) is not just the Schwarz inequality; it is

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precise than that of Bermant (in which V_2^* is replaced by the area of a larger domain).

[5] H. Grötzsch, "Über einige Extremalprobleme der korformen Abbildung I," *Berichte Akad. Leipzig*, 80(1928), 367-376. The inequality given by Grötzsch is a discontinuous version.

[6] L. Ahlfors, "Untersuchungen zur Theorie der korformen Abbildung und der ganzen Funktionen," *Acta Soc. sci. fenn.*, N.S. 1, 9(1930).

the Schwarz inequality plus a geometrical idea, and it is the latter which makes it important.

We now deduce the theorem, and suppose that $f \neq e^{i\alpha}z$. We write TG for the transform of a set G by f. We apply (2.2) to the mapping $\lg T\{\phi(r)\} \sim \lg \phi(r)$, with $G_1 = \lg T\phi(r)$, $G_2 = \lg\phi(r)$. Let d_r be the minimum distance of $T\{|z| = r\}$ from 0, and write $\lg w = \xi + i\eta$. We take for $q(\eta)$ all points of $\lg\{W^* \times G_1\}$ which lie over the line $\eta =$ constant, $0 \leq \eta < 2\pi$, so that $mH = 2\pi$. Then $q(\eta)$ connects all the boundary elements whose projection on the w-plane is intersected by the line $\eta =$ constant. Since the projection of $l(\eta)$ on the real axis has length $\lg \frac{1}{r}$, $L(\eta) \geq \lg \frac{1}{r} = \lambda$. Moreover, since $f \neq e^{i\alpha}z$, we have, for all $r < r_0$ and $\eta \in h$,

$$L_r(\eta) \geq \lg \frac{1}{r} + \delta, \tag{2.3}$$

where h and δ are independent of r, and $mh > 0$, $\delta > 0$. If now we make use of (2.3) in line (2.1), we obtain the following inequality (slightly stronger than (2.2))

$$\left(2\pi \lg \frac{1}{r}\right)^2 + \epsilon \lg \frac{1}{r} \leq V_1 \cdot V_2 \qquad (r < r_0) \tag{2.4}$$

where ϵ is independent of r, and $\epsilon > 0$.

Let $n^*(w)$ be the number of times the point w is covered by W^*. Then, writing $W^*(r)$ for that portion of W^* which lies outside the transform of $|z| = r$ by f, and writing $w = Re^{i\Phi}$, we have

$$\begin{aligned} V_1 &= \iint_{W^*(r)} \left|\frac{1}{w}\right|^2 R\,dR\,d\Phi \leq \int_{d_r}^{D}\int_{-\pi}^{\pi} \frac{n^*(Re^{i\Phi})}{R^2} R\,dR\,d\Phi \\ &= \int_{d_r}^{D} p^*(R)\,d(2\pi \lg R), \end{aligned} \tag{2.5}$$

where $p^*(R) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (n^*(Re^{i\Phi})\,d\Phi$. Let $A(d_r, R)$ be the area of the portion of W^* lying in the annulus $d_r < |w| < R$. Then

$$\begin{aligned} A(d_r, R) &= \int_{d_r}^{R} p^*(\underline{R})\,d(\pi\underline{R}^2) \\ &= \int_0^R p^*(\underline{R})\,d(\pi\underline{R}^2) - \int_0^{d_r} p^*(\underline{R})\,d(\pi\underline{R}^2) \\ &\leq \pi R^2 - \pi d_r^2 \end{aligned} \tag{2.6}$$

for all $R > d_r$, if r (and so d_r) is small enough. In fact, the hypothesis of mean valency 1 asserts that

$$\int_0^R p^*(\underline{R})\, d(\pi \underline{R}^2) \leq \pi R^2 \tag{2.7}$$

for all $R > 0$. Since $f(0) = 0$, there exists a circle of positive radius and center $w = 0$ every point P of which is covered at least once by W^*. Since, by (2.7), it is covered on the average at most once, it follows that the circle is covered exactly once by W^*. Hence, if d_r is sufficiently small,

$$\int_0^{d_r} p^*(R)\, d(\pi R^2) = \pi d_r^2; \tag{2.8}$$

and (2.7) and (2.8) together give (2.6). Finally by (2.5) we have, integrating by parts,

$$V_1 \leq \int_{d_r}^D p^*(R)\, d(2\pi \lg R) = \left[\frac{1}{R^2}\int_{d_r}^R p^*(\underline{R})\, d(\pi \underline{R}^2)\right]_{d_r}^D + 2\int_{d_r}^D \left(\int_{d_r}^R p^*(\underline{R})\, d(\pi \underline{R}^2)\right)\frac{dR}{R^3}$$

$$\leq \pi \frac{D^2 - d_r^2}{D^2} + 2\int_{d_r}^D (\pi R^2 - \pi d_r^2)\frac{dR}{R^3} = 2\pi \lg \frac{D}{d_r}$$

by (2.6). Therefore

$$V_1 \leq 2\pi \lg \frac{D}{d_r} \tag{2.9}$$

for all sufficiently small r.

Substituting from (2.9) into (2.4) we obtain, since $V_2 \leq 2\pi \lg \frac{1}{r}$,

$$\left(2\pi \lg \frac{1}{r}\right)^2 + \epsilon \lg \frac{1}{r} \leq \left(2\pi \lg \frac{D}{d_r}\right)\left(2\pi \lg \frac{1}{r}\right).$$

If ϵ_1 is chosen small enough, we then have, for $0 < r < r_0$,

$$2\pi \lg \frac{1}{r} + \epsilon_1 \leq \sqrt{\left(2\pi \lg \frac{1}{r}\right)^2 + \epsilon \lg \frac{1}{r}} \leq \sqrt{\left(2\pi \lg \frac{D}{d_r}\right)\left(2\pi \lg \frac{1}{r}\right)}$$

$$\leq \frac{2\pi \lg \frac{D}{d_r} + 2\pi \lg \frac{1}{r}}{2}$$

by the inequality of Schwarz; that is

$$\lg D + \lg \frac{r}{d_r} \geq \frac{\epsilon_1}{2\pi} > 0,$$

where ϵ_1 is independent of r. Finally, letting $r \to 0$, $\frac{r}{d_r} \to 1$, and we obtain the theorem.

The hypothesis (weaker than mean valency 1) that

$$\overline{\lim_{R \to 0}} \frac{1}{2\pi \lg \frac{D}{R}} \int_R^D p^*(R)\, d(2\pi \lg R) \leq 1$$

is all that we have used, and is therefore sufficient for the truth of the theorem.

3. Many applications of the same sort could be given, but without introducing anything essentially new, and I shall therefore merely state a final example.

Let S_k, $(k = 1, 2, \cdots, \nu)$, be doubly-connected domains wound round the circle $|z| = r$, and lying in the annulus $r < |z| < 1$. Suppose that S_k is mapped on the annulus $r_k < |z| < 1$; then r_k (dependent on S_k) is the modulus of S_k. Let B_1 be the "inner" boundary of S_k corresponding to $|z| = r_k$, and let B_2 be the boundary corresponding to $|z| = 1$. Let S_k^* be the set of points of S_k seen both from 0 if B_2 is opaque and B_1 transparent, and from ∞ looking toward 0 if B_1 is opaque and B_2 transparent. (Here (as above) it is understood that a boundary obstructs vision only for the sheet which it bounds). We write

$$p^*(\rho) = \sum_{(k)} p_k^*(\rho),$$

where $p_k^*(\rho)$ is the valency function (defined above) corresponding to S_k^*.

THEOREM 2. *If,*

$$\int_r^1 p^*(\rho)\, d(\lg \rho) \leq p \lg \frac{1}{r} \tag{3.1}$$

where $p > 0$ (logarithmic mean p-valency), then

$$r^p \leq \prod_{(k)} r_k$$

and there is equality if, and only if, there is equality in (4.1) *and each S_k as an annulus.*

The result of the theorem (with $p = 1$) is a known result of Koebe when each S_k is schlicht and no two S_k overlap. By omitting the valency hypothesis (3.1) altogether, we have as a final result:

THEOREM 3.

$$r^p \leq \prod_{k=1}^{p} r_k, \tag{3.2}$$

and there is equality only if each S_k is the annulus $r < |z| < 1$.[7]

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[7] As a final remark it is perhaps worthwhile to add that the above methods may sometimes be used to short-circuit arguments which depend on subordination (in the Lindelöf sense) to schlicht functions. For example, suppose that $f(z) = a_1 z + \cdots$ is regular in $|z| < 1$, and that $f(z_1) f(z_2) \neq 1$ for $|z_1| < 1$, $|z_2| < 1$. Then, if Γ is the boundary of W^* (the portion of the transform of $|z| < 1$ by f which can be seen from $f(0)$), we have from the lemma that

$$2\pi \lg |a_1| \leq \int_\Gamma \lg R \, d\Phi \leq 0.$$

That is, $|a_1| \leq 1$. This is the theorem of Bieberbach-Eilenberg. See e.g. W. Rogosinski, "On a theorem of Bierberbach-Eilenberg," *Journ. Lond. Math. Soc.*, 14(1939), 4–11 (where the general result is deduced from the special case when f is schlicht by a "subordination" argument).

ON DISTORTION IN ANALYTIC TRANSFORMATIONS

D. C. SPENCER

1. Suppose that

$$f(z) = a_1 z + a_2 z^2 + \cdots$$

is regular in $|z| < 1$, and maps the unit circle $|z| < 1$ on a Riemann domain W. Let $n(w)$ be the number of times W covers the point w. Then if $n(w) \leq 1$ for all w, and if d is the distance of the boundary of W from $w = 0$, it is well-known that

$$d \geq \frac{|a_1|}{4}. \tag{1.1}$$

In particular, if C is any regular curve connecting $z = 0$ to a point of $|z| = 1$, then

$$\int_C |f'(z)| \cdot |dz| \geq \frac{|a_1|}{4},$$

and so the average distortion cannot be too small.

2. By relaxing both hypothesis and conclusion we obtain the following more general theorem of the same type:

THEOREM 1. *Suppose that*

$$f(z) = a_1 z + a_2 z^2 + \cdots$$

is regular for $|z| < 1$, *and that* $n(0) = 1$. *Let*

$$d = \operatorname*{Inf}_{|z|<1} \left| \frac{f(z)}{z} \right|.$$

Then if there is a real number β *for which* $n(de^{i\beta}) = 0$, *we have*

$$d > \frac{|a_1|}{13}. \tag{2.1}$$

The result of Theorem 1 is not best possible, and the problem of the best constant on the right-hand side of (2.1) remains open. If $n(de^{i\beta}) > 0$ for all β, the theorem is false as is shown by the function $f(z) = z - z^2$.

It is possible to deduce the following theorem from Theorem 1:

THEOREM 2. *Suppose that $f(z)$ satisfies the hypotheses of Theorem 1, and that, in addition, $|f(z)| < M$ for $|z| < 1$. Then, if $0 < \rho < 1$,*

$$|\arg f(\rho e^{i\psi})| \leq B \log \frac{1}{1-\rho} + O(1),$$

where

$$B = \frac{1}{\pi} \log \frac{13M}{|a_1|}.$$

The constant in the $O(1)$ depends on M, $|a_1|$, and the particular determination of arg f chosen.

3. Given a Riemann domain W, we write $w = Re^{i\psi}$, and define

$$p(R) = \frac{1}{2\pi} \int_{-\pi}^{\pi} n(Re^{i\psi})\, d\psi.$$

If now

$$\int_0^R p(R)\, d(\pi R^2) \leq \pi R^2$$

for all $R > 0$, we say that W is mean 1-valent. If the stronger hypothesis is satisfied that

$$\int_0^R n(Re^{i\psi})\, d(\pi R^2) \leq \pi R^2 \qquad (R > 0)$$

uniformly in ψ, we say that W is strongly mean 1-valent.

If $f(z)$ is strongly mean 1-valent, then (1.1) is true, and there is equality only if $f(z) = \dfrac{|a_1| \cdot |z|}{(1+\gamma z)^2}$, $|\gamma| = 1$.* Now the result

* This result was wrongly stated for the family of mean 1-valent functions in the *Bulletin of the Amer. Math. Soc.*, 46 (1940), Abstract 126, p. 55, as was the allied result

$$|a_1||z|/(1+|z|)^2 \leq |f(z)| \leq |a_1||z|/(1-|z|)^2.$$

A weaker version of the latter result has also been proved for mean 1-valent functions; namely that

$$A_1|a_1||z|/(1+|z|)^2 \leq |f(z)| \leq A_2|a_1||z|/(1-|z|)^2,$$

where A_1 is somewhat less, A_2 somewhat greater, than unity. For the upper bound see D. C. Spencer, "On finitely mean valent functions II," *Trans. Amer. Math. Soc.*, 48(1940), 418–435.

$$|a_2| \leq 2|a_1|$$

$\left(\text{with equality only for } f = \dfrac{a_1 z}{(1+\gamma z)^2}, |\gamma| = 1\right)$ is true for the family of mean 1-valent functions, and the question arises whether (1.1) is not also true for this larger family. Although I am not able to answer this question at present, I should like to make a few remarks concerning it.

First, mean 1-valency implies that $n(0) = 1$, and that there exists a β for which $n(de^{i\beta}) = 0$. Therefore, by Theorem 1, we have that

$$d > \frac{|a_1|}{A} \tag{3.1}$$

where, in fact, $A = 13$. The problem, then, is to determine the best constant possible. The value 13 for A is much too large, and it is easy to show that (3.1) is true with an A of approximately half this size. Much more detailed computations have been made, however, which indicate (though they do not prove) that (3.1) is still true with an A only slightly greater than 4. Lack of space prevents me from giving a detailed description of this work here; but the method involves comparison of W (the map of $|z| < 1$ by f) with the slit plane $\left(\text{the map of } |z| < 1 \text{ by } K = \dfrac{a_1 z}{(1+z)^2}\right)$, and differs from classical methods in that only a portion of W is used.

Detailed proofs of all results stated above will appear shortly.

Massachusetts Institute of Technology.

TABLES OF COMPLETE ELLIPTIC INTEGRALS

By Carl Heuman

1. Introduction

The present paper contains a set of tables of complete elliptic integrals computed and collected especially for applications to certain dynamical problems.

The tabulated functions are four in number and are denoted by $F_0(\alpha)$, $G_0(\alpha)$, $E_0(\alpha)$, and $\Lambda_0(\alpha, \beta)$ respectively. The definitions of these functions and their connections with the functions of Legendre will be discussed in the following. Before doing this we make here some preliminary remarks.

The values of the functions in the tables lie all in the interval 0 to 5 and are always given with six decimals. The integral part and the first decimal will in general be common for all columns in several rows. In the order to save space, we have then separated these figures and have entered them only in the first column of the first row. When the first decimal has to be changed inside a row, this is marked either by a *dot* or by a *dash* over the first figure in the remaining columns. In the former case we have to *increase*, in the latter to *diminish* the first decimal by one unit. Thus, for example, we find for

$$\alpha = 87.2^\circ,\ F_0(\alpha) = 2.7\ \dot{0}5796, \text{ which means } 2.8\ 05796,$$

$$\alpha = 0.4^\circ,\ E_0(\alpha) = 1.0\ \bar{9}9988, \text{ which means } 0.9\ 99988.$$

The same marks are used in similar sense in the tables of differences. Thus on page 150 we find the difference of $F_0(\alpha)$ for $\alpha = 64.2^\circ$ listed as 20 $\dot{0}0$, which means 21 00, and on page ~~966~~ the difference of $E_0(\alpha)$ for $\alpha = 75.4^\circ$ listed as 5 $\bar{9}7$, which means 4 97, etc.

page 158

The values of the tables have been computed with the use of two tables in Legendre's *Traité des fonctions elliptiques* (3 vol., Paris 1825–28), viz., Table I (vol. 2, pages 222–243) and Table IX (ibid., pages 292–363). The computation led to the discovery of some misprints in these tables; a list of these is given in the following on pages 143–144.

We now pass on to an account of the individual functions.

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2. The functions $F_0(\alpha)$ and $E_0(\alpha)$

The elliptic normal integrals of the first and the second kind, introduced by Legendre, are functions of two variables, the *modular angle* α, located in the interval $(0, \pi/2)$, and the *amplitude* φ. The former determines the *modulus* $k = \sin \alpha$. If we put

$$\Delta\psi = [1 - k^2 \sin^2 \psi]^{\frac{1}{2}},$$

these integrals are

$$F(\alpha, \varphi) = \int_0^{\varphi} \frac{d\psi}{\Delta\psi}, \qquad E(\alpha, \varphi) = \int_0^{\varphi} \Delta\psi \, d\psi.$$

When $\varphi = \pi/2$, the integrals are called *complete* and may be denoted by $F(\alpha)$ and $E(\alpha)$, respectively.

The functions $F_0(\alpha)$ and $E_0(\alpha)$, tabulated on pages 146–151 and 154–159, are connected with these by the relations

$$F_0(\alpha) = \frac{2}{\pi} F(\alpha), \qquad E_0(\alpha) = \frac{2}{\pi} E(\alpha).$$

By this modification a certain simplification is attained in the uses here intended.

The derivatives of these functions are

$$\frac{dF_0}{d\alpha} = \frac{1}{kk'} (E_0 - k'^2 F_0), \qquad \frac{dE_0}{d\alpha} = -\frac{k'}{k} (F_0 - E_0).$$

Here k' denotes the "complementary" modulus

$$k' = \cos \alpha = [1 - k^2]^{\frac{1}{2}},$$

further $F_0 = F_0(\alpha)$ and $E_0 = E_0(\alpha)$.

We have also

$$\frac{\pi}{2} \frac{dF_0}{d\alpha} = \frac{k}{k'} \int_0^{\pi/2} \frac{\cos^2 \psi}{\Delta\psi} \, d\psi, \qquad \frac{\pi}{2} \frac{dE_0}{d\alpha} = -kk' \int_0^{\pi/2} \frac{\sin^2 \psi}{\Delta\psi} \, d\psi,$$

from which it is apparent that the former derivative is $\geqq 0$, the latter $\leqq 0$. $F_0(\alpha)$ is consequently an increasing, $E_0(\alpha)$ a decreasing function. When α increases from 0 to $\pi/2$, $F_0(\alpha)$ increases from 1 to ∞, while $E_0(\alpha)$ decreases from 1 to $2/\pi \sim 0.636620$. At first, the functions vary very slowly, as is seen from the series expansions, derived by Legendre,

$$\begin{cases} F_0(\alpha) = 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \cdots, \\ E_0(\alpha) = 1 - \left(\frac{1}{2}\right)^2 \frac{k^2}{1} - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{k^4}{3} - \cdots. \end{cases}$$

The tables give the values of these functions to six decimals for every tenth of a degree of α, from 0.0° to 89.9°. They have been computed from Legendre's Table I, which for the same arguments gives $^{10}\log F(\alpha)$ and $^{10}\log E(\alpha)$ to 12 or 14 decimals. After the addition of $^{10}\log (2/\pi)$ to the logarithms, the corresponding numbers have been determined to 10 places from Vega's *Thesaurus Logarithmorum*. The results have been checked by the forming of differences and then abbreviated to six decimals.

3. Interpolation in the tables for $F_0(\alpha)$ and $E_0(\alpha)$

If the argument α is not an integral multiple of 0.1°, the value of the function $F_0(\alpha)$ or $E_0(\alpha)$ can conveniently be determined by interpolation in the proper table, except in a certain portion of the table for $F_0(\alpha)$. The interpolation in this table becomes in fact difficult if $\alpha > 65°$, and quite impracticable for α near to 90°. The required value can then be found by the use of a certain auxiliary function $G_0(\alpha)$ which will be discussed in section 4.

In order to simplify interpolation, the tables also contain the first differences δ_1 of the functional values, for $F_0(\alpha)$ within the range $\alpha =$ 0° to 65°, for $E_0(\alpha)$ in the whole range 0° to 90°. For $F_0(\alpha)$ these differences are positive and increase with α, from zero to very great values. For $E_0(\alpha)$, on the other hand, they are negative; their numerical value increases from zero to a certain maximum, and then diminishes again to zero. The argument corresponding to this maximum coincides practically with the value α_*, for which the derivative

$$\frac{d^2E_0}{d\alpha^2} = \frac{1}{k^2}\{F_0 - (1 + k^2)E_0\}$$

becomes = 0. Approximately $\alpha_* = 58.5°$.

From the differences δ_1 thus tabulated we easily obtain the second differences δ_2, which we suppose expressed in units of the last decimal. The influence of this difference is not noticeable as long as the numerical value $|\,\delta_2\,|$ is $\leqq 4$. This is the case in the interval (0°, 46°) for $F_0(\alpha)$ and (0°, 85°) for $E_0(\alpha)$. Thus in these intervals linear interpolation may be used and gives as great accuracy as the tables allow. In the ensuing intervals, (46°, 65°) for $F_0(\alpha)$ and (85°, 90°) for $E_0(\alpha)$, δ_2 is positive and may sometimes be greater than 4 but is always less than 12. As a consequence we find that the value obtained by linear interpolation will then occasionally be a little too large. If $\alpha = (\text{A} + \rho)\cdot 0.1°$, where A is an integer and $0 < \rho < 1$, we have to diminish the last decimal of the obtained value by $0.5\,\rho(1 - \rho)\delta_2$ units. Since $\delta_2 < 12$, this correction can not exceed $1\frac{1}{2}$ units.

4. The function $G_0(\alpha)$

As already mentioned, $G_0(\alpha)$ is an auxiliary function introduced in order to facilitate the determination of the function $F_0(\alpha)$ for arguments α in the interval (65°, 90°), which are not integral multiples of 0.1°.

The function in question is defined by the relation

$$F_0(\alpha) = G_0(\alpha) - a \cdot {}^{10}\log (90 - \alpha^\circ),$$

where

$$a = \frac{2}{\pi} \cdot {}^{\circ}\log 10 \sim 1.465871.$$

When α grows from 65° to 90°, $G_0(\alpha)$ decreases from ~ 3.519019 to $\frac{2}{\pi} \cdot {}^{\circ}\log \frac{720}{\pi} \sim 3.459724$.

The table on pages 152 and 153 gives the value of $G_0(\alpha)$ for every tenth of a degree of α from 65.0° to 89.9°, as well as the corresponding differences. For an intermediate value of α the corresponding value of $G_0(\alpha)$ can easily be determined, since linear interpolation can be used within the whole range. We have then to take ${}^{10}\log (90 - \alpha^\circ)$ from an ordinary six or seven place table of logarithms, multiply this logarithm by the constant a and subtract the result from $G_0(\alpha)$, and the required value of $F_0(\alpha)$ is obtained. If a calculating machine is available for these operations, the procedure is very simple. A couple of examples illustrating the computation of $F_0(\alpha)$ by this method are inserted at the top of pages 152 and 153.

The basis for the introduction of this auxiliary function lies in the following relation, due to Legendre, viz.,

$$F(\alpha) = F_0(\alpha') \cdot {}^{\circ}\log \frac{4}{k'} - P(k'),$$

where $\alpha' = \pi/2 - \alpha$, $k' = \sin \alpha' = \cos \alpha$, and

$$P(k') = \left(\frac{1}{2}\right)^2 k'^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \left(1 + \frac{2}{3 \cdot 4}\right) k'^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \left(1 + \frac{2}{3 \cdot 4} + \frac{2}{5 \cdot 6}\right) k'^6 + \cdots.$$

We obtain from this

$$G_0(\alpha) = F_0(\alpha) + \frac{2}{\pi} {}^{\circ}\log \left\{\frac{180}{\pi} \operatorname{arc\,sin} k'\right\}$$

$$= \frac{2}{\pi} \left\{ {}^{\circ}\log \frac{720}{\pi} - P(k') + {}^{\circ}\log \frac{\operatorname{arc\,sin} k'}{k'} - [F_0(\alpha') - 1] \cdot {}^{\circ}\log \frac{k'}{4} \right\}.$$

When k' tends to zero, the last term within the brackets tends to zero, since

$$F_0(\alpha') - 1 = \left(\frac{1}{2}\right)^2 k'^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k'^4 + \cdots.$$

The two preceding terms also tend to zero, and we see then that $G_0(\alpha)$ tends to $\frac{2}{\pi}\log\frac{720}{\pi}$, when α tends to 90°.

The values of $G_0(\alpha)$ in the tables have been computed from the corresponding values of $F_0(\alpha)$, originally with 10 decimals. The logarithms of $(90 - \alpha°)$ have been taken from Vega's Thesaurus.

5. Integrals of the third kind

Legendre's normal integral of the third kind is

$$\Pi(\alpha, p, \varphi) = \int_0^{\varphi} \frac{1}{1 - p\sin^2\psi}\cdot\frac{d\psi}{\Delta\psi}.$$

It depends thus upon a third variable, the *parameter* p, in addition to the modular angle α and the amplitude φ.

The integrals of this kind fall into two classes, according as the function

$$\tau(p) = p(p - k^2)(p - 1)$$

is negative or positive. If $\tau(p) = 0$, i.e., if $p = 0$, k^2 or 1, then the integral $\Pi(\alpha, p, \varphi)$ can be expressed by integrals of the first and the second kind, together with elementary functions.

These two classes may be called the *circular* and the *hyperbolic*. For the former $\tau(p)$ is < 0, so that either $k^2 < p < 1$ or $p < 0$. For the latter $\tau(p)$ is > 0, so that either $0 < p < k^2$ or $p > 1$.

It is however a very remarkable fact that the elliptic integrals which occur in dynamical problems almost always belong to only one of these classes, viz., to the circular one. For this reason we shall here deal only with the circular case.

We have, therefore, to consider integrals whose parameter p is either negative or lies in the interval k^2 to 1. The former case can, however, be reduced to the latter. For if $p_1 < 0$, we put

$$\frac{p_1 - k^2}{p_1 - 1} = p,$$

and then $k^2 < p < 1$. Now, if $\mathfrak{q}_1$, $\mathfrak{q}$, $\mathfrak{p}$ are the positive square roots of

$$\mathfrak{q}_1^2 = (1 - p_1)\left(1 - \frac{k^2}{p_1}\right), \qquad \mathfrak{q}^2 = (1 - p)\left(1 - \frac{k^2}{p}\right), \qquad \mathfrak{p}^2 = -pp_1,$$

we can easily verify the following relation:

$$\mathfrak{q}_1\Pi(\alpha, p_1, \varphi) = \mathfrak{q}\Pi(\alpha, p, \varphi) + \frac{k^2}{\mathfrak{p}} F(\alpha, \varphi) + \arctan \frac{\mathfrak{p} \sin\varphi \cos\varphi}{\Delta\varphi}.$$

The integral $\Pi(\alpha, p_1, \varphi)$ with negative parameter can in this way be expressed by means of the integral $\Pi(\alpha, p, \varphi)$, where $k^2 < p < 1$. By virtue of this we can confine ourselves to considering integrals whose parameters lie in the interval k^2 to 1.

In the pendulum and gyroscope theory there occur often such integrals $\Pi(\alpha, p, \varphi)$ of the circular class and then usually in combination with the same factor $\mathfrak{q}$ as in the formula above. For this reason we gain a considerable simplification by introducing the product $\mathfrak{q}\Pi(\alpha, p, \varphi)$ as the normal integral instead of $\Pi(\alpha, p, \varphi)$ itself.

Let $\Pi(\alpha, p, \varphi)$ be an integral whose parameter p satisfies the condition $k^2 \leqq p \leqq 1$. Then there exists a unique angle β in the interval 0 to $\pi/2$ for which

$$\sin^2\beta = \frac{p - k^2}{k'^2 p}, \qquad \cos^2\beta = \frac{k^2(1-p)}{k'^2 p}.$$

With the notation

$$\Delta'\beta = [1 - k'^2 \sin^2\beta]^{\frac{1}{2}},$$

we obtain

$$p = \left[\frac{k}{\Delta'\beta}\right]^2, \qquad p - k^2 = \left[\frac{kk' \sin\beta}{\Delta'\beta}\right]^2, \qquad 1 - p = \left[\frac{k' \cos\beta}{\Delta'\beta}\right]^2,$$

$$\mathfrak{q} = \frac{k'^2 \sin\beta \cos\beta}{\Delta'\beta}$$

and

$$1 - p \sin^2\psi = \frac{k'^2 \cos^2\beta + k^2 \cos^2\psi}{(\Delta'\beta)^2}.$$

Hence $\mathfrak{q}\Pi(\alpha, p, \varphi) = \Lambda(\alpha, \beta, \varphi)$, where

$$\Lambda(\alpha, \beta, \varphi) = \int_0^{\varphi} \frac{k'^2 \sin\beta \cos\beta \, \Delta'\beta}{k'^2 \cos^2\beta + k^2 \cos^2\psi} \cdot \frac{d\psi}{\Delta\psi}. \tag{1}$$

This integral $\Lambda(\alpha, \beta, \varphi)$ can with advantage be used as the normal integral in many dynamical problems. It is a function of three variables, the *modular angle* α, the *parametric angle* β, and the *amplitude* φ.

6. The function $\Lambda_0(\alpha, \beta)$

The corresponding complete integral, which we get for $\varphi = \pi/2$, may be denoted by $\Lambda(\alpha, \beta)$. The function tabulated on pages 160–197 is connected with $\Lambda(\alpha, \beta)$ by the equation

$$\Lambda_0(\alpha, \beta) = \frac{2}{\pi}\Lambda(\alpha, \beta).$$

This is then the same modification that we have used with integrals of the first and second kinds, and the motive is the same as in that case.

Legendre has proved that every complete integral of the third kind is expressible in terms of integrals of the first and the second kind, possibly together with elementary functions. This must then be true also for $\Lambda_0(\alpha, \beta)$. Indeed, the expression for it has quite a simple form:

$$\Lambda_0(\alpha, \beta) = F_0(\alpha)E(\alpha', \beta) - \{F_0(\alpha) - E_0(\alpha)\}\cdot F(\alpha', \beta), \qquad (2)$$

where, as above, $\alpha' = \pi/2 - \alpha$.

By this formula $\Lambda_0(\alpha, \beta)$ is directly determined as a function of the two independent variables, α and β, within the region $0 \leqq \alpha \leqq \pi/2$, $0 \leqq \beta \leqq \pi/2$. For $\alpha = \pi/2$, however, a limiting passage is necessary, and we find that the function then is $= (2/\pi)\cdot\beta$. For $\beta = \pi/2$ the function becomes $= 1$, according to the relation

$$F_0(\alpha)E(\alpha') + E_0(\alpha)F(\alpha') - F_0(\alpha)F(\alpha') = 1, \qquad (3)$$

which was derived by Legendre. For the extreme values, 0 and $\pi/2$, of the variables, we have consequently

$$\Lambda_0(0, \beta) = \sin\beta, \qquad \Lambda_0\left(\frac{\pi}{2}, \beta\right) = \frac{2}{\pi}\beta,$$

$$\Lambda_0(\alpha, 0) = 0, \qquad \Lambda_0\left(\alpha, \frac{\pi}{2}\right) = 1.$$

The partial derivatives of the function $\Lambda_0(\alpha, \beta)$ are

$$\frac{\partial\Lambda_0}{\partial\alpha} = -\frac{k'}{k}\cdot\frac{\sin\beta\cos\beta}{\Delta'\beta}\cdot(F_0 - E_0), \qquad \frac{\partial\Lambda_0}{\partial\beta} = \frac{1}{\Delta'\beta}\cdot(E_0 - k'^2\sin^2\beta\, F_0),$$

where $F_0 = F_0(\alpha)$, $E_0 = E_0(\alpha)$. The former derivative is zero at the borders but negative in the interior of the region in question; the latter is zero for $(\alpha = 0, \beta = \pi/2)$ but is positive elsewhere. If β is constant, $0 < \beta < \pi/2$, while α increases from 0 to $\pi/2$, the function $\Lambda_0(\alpha, \beta)$ consequently decreases from $\sin\beta$ to $(2/\pi)\cdot\beta$. If α is constant, $0 \leqq$

$\alpha \leqq \pi/2$, while β increases from 0 to $\pi/2$, the function increases from 0 to 1. For the whole region we have then $0 \leqq \Lambda_0(\alpha, \beta) \leqq 1$.

The tables on pages 160–195 give the values of the function $\Lambda_0(\alpha, \beta)$ to 6 decimals for every integral degree of α and β, from 0° to 90°. The values have been computed from the formula (2) with aid of Table IX in Legendre's *Traité*. This table gives $F(\alpha, \varphi)$ and $E(\alpha, \varphi)$ to 9 or 10 decimals for integral degrees of α and φ, and has thus furnished the values of $F(\alpha', \beta)$ and $E(\alpha', \beta)$ occurring in (2). The results, as a rule computed to 9 decimals, have been checked by the formation of differences and then abbreviated to 6 places.

The supplementary table on pages 196–197 will be mentioned below (§10).

7. Expression for $\Lambda_0(\alpha, \beta)$ by means of a series

In the integrals

$$F(\alpha', \beta) = \int_0^\beta \frac{d\psi}{\Delta'\psi}, \qquad E(\alpha', \beta) = \int_0^\beta \Delta'\psi \, d\psi,$$

where

$$\Delta'\psi = [1 - k'^2 \sin^2 \psi]^{\frac{1}{2}},$$

we can expand $1/\Delta'\psi$ and $\Delta'\psi$ in powers of $k' \sin \psi$ and then integrate term by term. If we insert the results in the formula (2), we get an expression of the form

$$\Lambda_0(\alpha, \beta) = a_0 b_0 - a_2 b_2 - a_4 b_4 - a_6 b_6 - \cdots, \tag{4}$$

where $a_0, a_2, a_4, \cdots$ are functions only of α, and $b_0, b_2, b_4, \cdots$ only of β. The former functions are

$$\begin{cases} a_0 = E_0, \qquad a_2 = \frac{1}{2}(2F_0 - E_0)k'^2, \qquad a_4 = \dfrac{1 \cdot 1}{2 \cdot 4}(4F_0 - 3E_0)k'^4, \\ a_6 = \dfrac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}(6F_0 - 5E_0)k'^6, \cdots \end{cases}$$

where, as before, $F_0 = F_0(\alpha)$ and $E_0 = E_0(\alpha)$. Further,

$$b_n = \int_0^\beta \sin^n \psi \, d\psi.$$

Hence

$$b_0 = \beta, \qquad b_2 = \tfrac{1}{2}(\beta - \sin \beta \cos \beta),$$

$$b_n = \frac{n-1}{n} b_{n-2} - \frac{1}{n} \sin^{n-1} \beta \cos \beta.$$

The formula (4) is suitable for direct computation of $\Lambda_0(\alpha, \beta)$, if the modular angle α is $\geqq 80°$. Indeed, in this case in order to obtain 6 correct decimals, we need take account of only a few terms, e.g. for $\alpha = 80°$, 2, 3 or 4 terms, according as β lies in the interval (0°, 18°), (18°, 50°) or (50°, 90°). These intervals are to be found in the following table, in which we also find the corresponding intervals for $\alpha = 81°$ to 89°; n is the requisite number of terms.

α	$n =$ 2	3	4		α	$n =$ 2	3	
80°	$\beta =$ 0°	18°	50°	90°	85°	$\beta =$ 0°	31°	90°
81	0	20	56	90	86	0	37	90
82	0	22	63	90	87	0	47	90
83	0	24	74	90	88	0	69	90
84	0	27	90		89	0	90	

8. Extension of the range for β

In some applications it is found convenient not to limit the parametric angle β to the interval $(0, \pi/2)$ but to let the formula (2) define the function $\Lambda_0(\alpha, \beta)$ for all real values of β.

From the relations

$$F(\alpha', -\beta) = -F(\alpha', \beta), \; F(\alpha', \pi \pm \beta) = 2F(\alpha') \pm F(\alpha', \beta),$$

$$E(\alpha', -\beta) = -E(\alpha', \beta), \; E(\alpha', \pi \pm \beta) = 2E(\alpha') \pm E(\alpha', \beta),$$

follows then, according to (3),

$$\Lambda_0(\alpha, -\beta) = -\Lambda_0(\alpha, \beta), \; \Lambda_0(\alpha, \pi \pm \beta) = 2 \pm \Lambda_0(\alpha, \beta).$$

More generally, for every integer n, we have

$$\Lambda_0(\alpha, n\pi \pm \beta) = 2n \pm \Lambda_0(\alpha, \beta)$$

and $\Lambda_0\left(\alpha, \frac{n\pi}{2}\right) = n.$

9. An addition-formula

Let β_1 and β_2 be two angles in the interval $(0, \pi/2)$. If we compute from them the angles γ_1 and γ_2 in the same interval by the formulas

$$\tan \gamma_1 = \tan \beta_1 \, \Delta' \beta_2 , \qquad \tan \gamma_2 = \tan \beta_2 \, \Delta' \beta_1 ,$$

and put $\gamma_1 + \gamma_2 = \beta_3$, then the following relation holds:

$$\cos \beta_3 = \cos \beta_1 \cos \beta_2 - \sin \beta_1 \sin \beta_2 \, \Delta' \beta_3 ,$$

and we have

$$F(\alpha', \beta_1) + F(\alpha', \beta_2) = F(\alpha', \beta_3),\ E(\alpha', \beta_1) + E(\alpha', \beta_2) = E(\alpha', \beta_3) + r,$$

where

$$r = k'^2 \sin\beta_1 \sin\beta_2 \sin\beta_3 .$$

Consequently we obtain from (2)

$$\Lambda_0(\alpha, \beta_1) + \Lambda_0(\alpha, \beta_2) = \Lambda_0(\alpha, \beta_3) + rF_0(\alpha).$$

We notice especially the case when $\beta_3 = \pi/2$. We get then the following result: If two angles β_1, β_2 in the interval $(0, \pi/2)$ are connected by the relation

$$\tan\beta_1 \tan\beta_2 = k^{-1}, \tag{5}$$

then

$$\Lambda_0(\alpha, \beta_1) + \Lambda_0(\alpha, \beta_2) = 1 + k'^2 \sin\beta_1 \sin\beta_2 F_0(\alpha). \tag{6}$$

If here $\beta_1 = \beta_2$ and the common value is denoted by β_0, then for this angle $\tan\beta_0 = k^{-1}$, and we get

$$2\Lambda_0(\alpha, \beta_0) = 1 + (1 - k)F_0(\alpha).$$

If the angles β_1 and β_2 in (5) are not equal, one of them must be $> \beta_0$, the other $< \beta_0$. By means of the formulas (5) and (6) it is then possible to pass from a greater parametric angle β_1 to a smaller one β_2, supposing that $\beta_1 > \beta_0$.

10. Interpolation in the table for $\Lambda_0(\alpha, \beta)$

Let α and β be two given angles within the interval $(0°, 90°)$, of which at least one is not an integral multiple of $1°$. The corresponding value of $\Lambda_0(\alpha, \beta)$ can then not be read directly in the table but must be determined through interpolation or in some other way. We shall show here, how one can proceed in order to obtain this value to 6 decimals, provided, however, that a small error, not exceeding two units of the sixth decimal, may be permitted.

If the given angles are expressed in degrees, three different cases may occur, viz.,

a) $\alpha = (\text{A} + \rho)°$, $\beta = \text{B}°$, b) $\alpha = \text{A}°$, $\beta = (\text{B} + \rho)°$,

c) $\alpha = (\text{A} + \rho)°$, $\beta = (\text{B} + \rho')°$.

Here A and B are integers, while $0 < \rho < 1$ and $0 < \rho' < 1$.

We shall, to begin with, consider the two first cases. We have then to interpolate, in case a) in a certain row, in case b) in a certain column in the table. In order to obtain the desired accuracy we must then, as a rule, use quadratic interpolation, but there are regions where linear interpolation is quite sufficient. On the other hand, there are also regions where quadratic interpolation is not sufficient, so that we have to resort to cubic interpolation or some other method.

How these different regions are distributed, is seen from the figures below. In these L denotes the regions for linear interpolation and Q those for quadratic interpolation, while T, A, B are particular regions for which special rules will be given below. Fig. 1 refers to case a), i.e

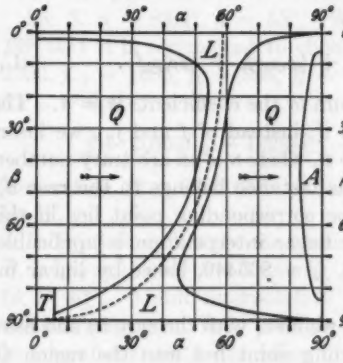

FIG. 1

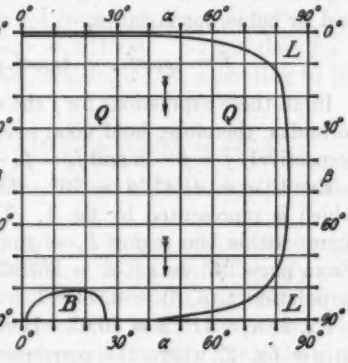

FIG. 2

interpolation in a horizontal row, fig. 2 to case b), interpolation in a vertical column. This is indicated by the directions of the arrows inserted.

In each interpolation the value sought is determined by several tabulated values. We can, for instance, proceed in the following way.

We express the values in units of the sixth decimal, putting in case a)

$$f = 10^6\Delta_\vartheta(\text{A} + \rho, \text{B}), \qquad f_n = 10^6\Delta_\vartheta(\text{A} + n, \text{B}),$$

and in case b)

$$f = 10^6\Delta_\vartheta(\text{A}, \text{B} + \rho), \qquad f_n = 10^6\Delta_\vartheta(\text{A}, \text{B} + n),$$

where n is an integer. The values f_n, corresponding to different n's, can then be taken from the table, provided that $\text{A} + n$, respectively $\text{B} + n$, is $\geqq 0$ and $\leqq 90$. In linear interpolation we use only two,

in quadratic three and in cubic four successive values f_n, which can be selected in different ways.

If we put $1 - \rho = \sigma$ and $n + \rho = \rho_n$, $n + \sigma = \sigma_n$, we have by linear interpolation

$$f = \sigma f_0 + \rho f_1 = f_0 + \rho(f_1 - f_0),$$

by quadratic

$$f = -\tfrac{1}{2}\rho\sigma f_{-1} + \rho_1\sigma f_0 + \tfrac{1}{2}\rho\rho_1 f_1, \qquad (Q_1)$$

or

$$f = \tfrac{1}{2}\sigma\sigma_1 f_0 + \rho\sigma_1 f_1 - \tfrac{1}{2}\rho\sigma f_2, \qquad (Q_2)$$

and by cubic, for instance,

$$f = \tfrac{1}{6}\rho\rho_1\sigma f_{-2} - \tfrac{1}{2}\rho\rho_2\sigma f_{-1} + \tfrac{1}{2}\rho_1\rho_2\sigma f_0 + \tfrac{1}{6}\rho\rho_1\rho_2 f_1. \qquad (C_1)$$

In all these expressions for f the sum of the coefficients is $= 1$. The formulas, therefore, hold good even if, instead of f and f_n, we insert respectively $\bar{f} = f - N$ and $\bar{f}_n = f_n - N$, where N is an arbitrary number.

Ex. 1. $\alpha = 41.4°$, $\beta = 70°$. This instance belongs to the case a), which is represented by fig. 1. The corresponding point lies in this figure within the region L, so that linear interpolation is applicable. From page 187 we get $f_0 = 867863$, $f_1 = 865449$, hence by linear interpolation $\Lambda_0(\alpha, \beta) = 0.866897$.

Ex. 2. $\alpha = 41°$, $\beta = 70.6°$. Here we meet with the case b) and have to use fig. 2, where the corresponding point lies into the region Q. Thus quadratic interpolation should be used. From the column $\alpha = 41°$ on page 233 it is seen, that it is convenient to choose $N = 860000$. We get then $\bar{f}_{-1} = 232$, $\bar{f}_0 = 7863$, $\bar{f}_1 = 15362$, and from this, according to (Q_1) for $\rho = 0.6$,

$$\bar{f} = -0.12\bar{f}_{-1} + 0.64\bar{f}_0 + 0.48\bar{f}_1 = 12378.2.$$

If we had used the formula (Q_2) instead, with $\bar{f}_2 = 22732$, we would have obtained

$$\bar{f} = 0.28\bar{f}_0 + 0.84\bar{f}_1 - 0.12\bar{f}_2 = 12377.9.$$

Both the formulas thus agree in giving $\Lambda_0(\alpha, \beta) = 0.872378$.

If $\alpha < 6°$ and $\beta \geqq 80°$, the representing point in fig. 1 falls into the rectangle denoted by T. In this domain interpolation in a row on page 164 is not convenient, because regions for linear, quadratic and cubic interpolation here run together. In order to simplify the practical use the accurate interpolations have been executed in the

rows $\beta = 80°, 81°, \cdots, 89°$ for every tenth of a degree of α in the interval (0°, 6°) and the results have been tabulated on pages 196–197. Thus for $\alpha = 5.4°$, $\beta = 81°$ we get directly from page 197 $\Lambda_0(\alpha, \beta) = 0.985663$. If the given argument α has two or more decimals, linear interpolation gives the value sought. Thus for $\alpha = 5.438°$, $\beta = 81°$ we have to diminish the value above by $0.38 \cdot 70 \sim 27$ units of the sixth decimal, so that $\Lambda_0(\alpha, \beta) = 0.985636$.

In fig. 1 there is also another region, requiring special care, which is denoted by A. Here α is $> 83°$, and $\Lambda_0(\alpha, \beta)$ can then easily be computed by the method exhibited in §7. In particular, this method should always be used if $\alpha > 89°$. In the remainder of the region cubic interpolation with the formula (C_1) gives the desired accuracy.

Ex. 3. $\alpha = 88.4°$, $\beta = 55°$. We see from the row $\beta = 55°$ on page 133 137 that it is convenient to choose $N = 611000$. We get then $\mathfrak{f}_{-2} = 2700$, $\mathfrak{f}_{-1} = 1685$, $\mathfrak{f}_0 = 884$, $\mathfrak{f}_1 = 336$, and from this, according to (C_1) for $\rho = 0.4$,

$$\mathfrak{f} = 0.056\mathfrak{f}_{-2} - 0.288\mathfrak{f}_{-1} + 1.008\mathfrak{f}_0 + 0.224\mathfrak{f}_1 = 632.3.$$

Hence $\Lambda_0(\alpha, \beta) = 0.611632$. The same value is obtained by the method in §7.

In fig. 2 there is only one special region. In this region, denoted by B, α has relatively small values ($1° < \alpha < 26°$) and β relatively great ($\beta > 80°$). Cubic interpolation with the formula (C_1) gives here the desired accuracy. It is possible to check the results by passing over to a smaller angle β, according to the formulas in §9. For if the given angle is denoted by β_1, the formula (5) determines a smaller angle β_2 and the point corresponding to (α, β_2) lies within Q. We are then able to determine more easily $\Lambda_0(\alpha, \beta_2)$, after which (6) gives $\Lambda_0(\alpha, \beta_1)$.

We have further to consider the case c), where neither α nor β is an integral multiple of 1°. It may suffice to treat an example.

Ex. 4. $\alpha = 41.4°$, $\beta = 70.6°$. The corresponding point lies in fig. 1 within L, in fig. 2 within Q. We begin then by computing $\Lambda_0(\alpha, \beta)$ for $\alpha = 41.4°$, $\beta = 69°$, 70° and 71°. This is done by linear interpolation in the three corresponding rows on page 187. We find

α	β	$\Lambda_0(\alpha, \beta)$
41.4°	69°	0.8 59233
"	70°	66897
"	71°	74431

From these three values we get then by quadratic interpolation the value sought, which corresponds to $\beta = 70.6°$. With $N = 859000$ we have

$\mathfrak{f}_{-1} = 233$, $\mathfrak{f}_0 = 7897$, $\mathfrak{f}_1 = 15431$, and from this, according to (Q_1) for $\rho = 0.6$, we get

$$\mathfrak{f} = -0.12\mathfrak{f}_{-1} + 0.64\mathfrak{f}_0 + 0.48\mathfrak{f}_1 = 12433.0.$$

Hence $\Lambda_0(\alpha, \beta) = 0.871433$.

In this way we can, as a rule, work out such a problem by several (3 to 5) interpolations, but the calculation may become rather laborious especially if the point, corresponding to (α, β), falls within A in fig. 1 or within B in fig. 2. In the former case, and generally if $\alpha > 80°$, direct computation, by means of the formula (4) in §7, is as a rule more convenient.

It remains to mention, how the limit curves in the figures have been constructed. This has been done by use of Newton's interpolation formula

$$f = f_0 + \rho\delta_1 - \tfrac{1}{2}\rho(1 - \rho)\delta_2 + \tfrac{1}{6}\rho(1 - \rho)(2 - \rho)\delta_3 - \cdots,$$

which can be applied both in case a) and in case b). In this f, f_0 and ρ have their former significance, while $\delta_1, \delta_2, \delta_3, \cdots$ are the successive differences, expressed like f and f_0 in units of the sixth decimal.

In the delimitation of the different regions in the figures the principle has now been, that within L, $|\delta_2|$ should be $\leqq 6$, within Q, $|\delta_2| > 6$ but $|\delta_3| \leqq 12$, and within A and B, $|\delta_3| > 12$. If we denote by d_n the numerical value of that term in the formula which contains the difference δ_n, we should then have within L that $d_2 \leqq |\delta_2 \div 8| \leqq 0.75$ and within Q that $d_3 \leqq |\delta_3 \div 9\sqrt{3}| < 0.77$ units of the sixth decimal. These values are, however, not exactly valid, because the limits thus determined have become smoothed out to continuous curves, but the deviations caused by this are small and of no importance.

By means of a supplementary investigation of the influence of the other terms we find then, that the desired accuracy is attained by linear interpolation within L and by quadratic within Q. For the regions T, A and B special investigations have been carried out, which led to the rules stated above.

In case a) the second difference δ_2 is approximately proportional to the derivative $\partial^2\Lambda_0/\partial\alpha^2$ and approaches zero simultaneously with it. The points (α, β) where this derivative is zero, must then belong to the region L (or T) in fig. 1. From the expression

$$\frac{\partial^2\Lambda_0}{\partial\alpha^2} = \frac{\sin\beta\cos\beta}{k^2(\Delta'\beta)^3} \cdot \varphi(\alpha, \beta),$$

where

$$\varphi(\alpha, \beta) = k'^2(E_0 - k'^2F_0)\sin^2\beta - \{(1 + k^2)E_0 - F_0\},$$

it follows that the locus for such points consists of the border lines $\beta = 0°$ and $\beta = 90°$ and further of a certain curve with the equation $\varphi(\alpha, \beta) = 0$. This is the dotted curve in fig. 1, which runs from the lower left corner ($\alpha = 0°, \beta = 90°$) to the point ($\alpha = \alpha_* \sim 58.5°, \beta = 0°$), α_* being the angle mentioned in §3 (page 129). We see that the region L contains this curve and connects the two horizontal border lines on which $\delta_2 = 0$.

The region L in fig. 2 is in a similar manner connected with the locus of the points where the derivative $\partial^2\Lambda_0/\partial\beta^2$ is $= 0$. This locus consists of the three border lines $\beta = 0°$, $\beta = 90°$ and $\alpha = 90°$, and we see that the region connects these lines.

In the preceding we have indicated the operations which are convenient for determining in different cases $\Lambda_0(\alpha, \beta)$ to 6 decimals with the error limit ± 0.000002. If only *four* decimals are required, the procedure is considerably simplified, inasmuch as linear interpolation is applicable both in rows and in columns within the entire domain. The error which then occurs is, as a rule, <0.000050, only exceptionally slightly greater, and at the utmost $= 0.000052$. This maximum occurs in case a) for $\alpha = 89.5°$, if β lies in the interval $(42°, 48°)$.

11. Remarks on the function $\Lambda(\alpha, \beta, \varphi)$

The incomplete integral $\Lambda(\alpha, \beta, \varphi)$ defined by the formula (1) in §5 is a function of the three arguments α, β and φ. We suppose that these arguments have given values, which are >0 and $<\pi/2$, and that one wishes to compute the corresponding value of $\Lambda(\alpha, \beta, \varphi)$. This can be done most simply by means of *theta-series*.

For this purpose we first determine $F(\alpha, \varphi)$, $F(\alpha', \beta)$, $F_0(\alpha)$, $F_0(\alpha')$ and $\Lambda_0(\alpha, \beta)$, where, as before, $\alpha' = \pi/2 - \alpha$. The first two can be obtained from Legendre's Table IX, the others from the present tables.

From these we then compute, firstly the Jacobi number q, secondly the two arguments x and y, according to the formulas

$$\log q = -\frac{F_0(\alpha')}{F_0(\alpha)}\cdot\pi, \qquad x = \frac{F(\alpha, \varphi)}{F_0(\alpha)}, \qquad y = \frac{F(\alpha', \beta)}{F_0(\alpha)}. \tag{7}$$

The theta-series here referred to are

$$\begin{cases} \mathfrak{s} = \sum_{n=1}^{\infty} 2q^{n^2} \sin 2nx \; sh \; 2ny, \\ \mathfrak{c} = 1 + \sum_{n=1}^{\infty} 2q^{n^2} \cos 2nx \; ch \; 2ny. \end{cases} \tag{8}$$

After we have computed $\mathfrak{s}$ and $\mathfrak{c}$ from these and thereafter w from the formula

$$\tan w = \frac{\mathfrak{s}}{\mathfrak{c}}, \qquad 0 < w < \frac{\pi}{2}, \tag{9}$$

we have

$$\Lambda(\alpha, \beta, \varphi) = x\Lambda_0(\alpha, \beta) - w. \tag{10}$$

Since $\beta < \pi/2$, $2y$ is $< {}^{\circ}\log q^{-1}$ and $2sh\,2ny < 2ch\,2ny < q^{-n} + 1$. If we denote by s_n and c_n the numerical values of those terms in the two series which contain the power q^{n^2}, both s_n and c_n will then be $< q^{n(n-1)} + q^{n^2}$. From this is seen that the series converge very rapidly for small values of q, so that we need take account of only a few terms. For $\alpha \leqq 45°$ q is $\leqq e^{-\pi} < 0.044$ and both s_2 and c_2 are $< 10^{-6}$.

For $\alpha > 45°$ the series (8) converge more slowly. The function w occurring in (10) can then conveniently be computed in the following way. In the expressions (8) we exchange q, x and y for q', x' and y' respectively, determined by the formulas

$${}^{\circ}\log q' = -\frac{F_0(\alpha)}{F_0(\alpha')} \cdot \pi, \qquad x' = \frac{F(\alpha', \beta)}{F_0(\alpha')}, \qquad y' = \frac{F(\alpha, \varphi)}{F_0(\alpha')}.$$

If $\mathfrak{s}'$, $\mathfrak{c}'$ denote the values of the series thus obtained and if w' is determined by the formula

$$\tan w' = \frac{\mathfrak{s}'}{\mathfrak{c}'}, \qquad 0 < w' < \frac{\pi}{2},$$

analogous to (9), then w and w' are connected by the relation

$$w + w' = \frac{2}{\pi} xx'.$$

For $\alpha > 45°$ q' is $< e^{-\pi}$ and the expressions for $\mathfrak{s}'$ and $\mathfrak{c}'$ converge very rapidly. We can then in this way conveniently compute w' and thereafter w from the relation just given.

The derivation of these formulas will be omitted here.

ERRATA

IN THE TABLES I AND IX BY LEGENDRE

The following list refers to the work mentioned on page 127: LEGENDRE, A. M., Traité des fonctions elliptiques, Tome 2, Paris 1826.

Page	Function	$\alpha°$	$\varphi°$	For	read
226	$\log F(\alpha)$	24.9	—	0.217 047 28···	0.217 047 38···
228	" "	30.9	—	8.228···	0.228···
"	$\log E(\alpha)$	34.4	—	0.157 376 108 8···	0.157 376 100 8···
242	" "	86.0	—	0.003 739 616 1···	0.003 739 613 1···
294	$E(\alpha, \varphi)$	4	51	0.88952···	0.88962···
296	"	10	5	0.00237 3···	0.08726 3···
298	"	8	90	1.56296···	1.56316···
302	"	15	54	0.93450···	0.93460···
310	"	22	78	1.31072···	1.31972···
322	"	35	45	0.76188···	0.76128···
324	"	42	27	0.46376···	0.46366···
326	"	45	73	1.13785 83···	1.13785 43···
330	"	"	"	"	"
352	"	79	32	0.53101 73···	0.53101 13···
293	$F(\alpha, \varphi)$	2	42	0.73311 099···	0.73311 009···
297	"	7	5	0.08729 8···	0.08726 8···
"	"	9	20	0.34293···	0.34923···
299	"	5	73	1.27598 584···	1.27598 564···
305	"	20	32	0.56174 72···	0.56174 52···
311	"	21	88	1.58584···	1.58784···
"	"	23	90	1.63631···	1.63651···
313	"	30	35	0.62203···	0.62003···
"	"	27	42	0.74754···	0.74574···
315	"	27	59	1.06251 39···	1.06251 29···
"	"	27	85	1.56840···	1.56480···
327	"	44	80	1.58906···	1.59606···
331	"	46	51	0.95158 7···	0.95157 7···
339	"	59	65	1.34195 7···	1.34196 7···
"	"	57	90	2.08035 816···	2.08035 806···
"	"	60	60	1.21253 6···	1.21259 6···
343	"	"	"	"	"
"	"	64	79	1.84693···	1.84793···
349	"	75	37	0.69130···	0.69190···
351	"	71	61	1.29179···	1.29719···
357	"	84	18	0.31936 7···	0.31939 7···
361	"	90	31	0.55956···	0.56956···
363	"	87	49	0.98238···	0.98328···
"	"	86	86	3.17204 1744	3.17030 9983[1]
"	"	87	87	3.45644 5172	3.45667 6096[1]

[1] The last decimals are uncertain.

Legendre has also published his tables of elliptic integrals in another, earlier work, viz.,

LEGENDRE, A. M., Exercices de Calcul intégral sur divers ordres de transcendantes et sur les quadratures. Tome 3. Paris 1816.

The greater part of the errors listed above occur also in this treatise and there are also some others. We note the following:

Function	$\alpha°$	$\varphi°$	For	read
$E(\alpha, \varphi)$	51	23	0.33502···	0.39502···
"	82	14	0.24195 9···	0.24196 9···
"	86	2	0.03489 9351	0.03489 9531

Legendre's Table IX has been reproduced photographically in a couple of recently published books, viz.,

POTIN, L., Formules et tables numériques. Paris 1925.

A. M. LEGENDRE, Tafeln der elliptischen Normalintegrale erster und zweiter Gattung. Herausgegeben von FRITZ EMDE. Stuttgart 1931.

In both cases it is the table in the earlier work (Exercices ···) that has been reproduced.

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	The function $F_0(\alpha)$						Differences				
		0	1	2	3	4	0	1	2	3	4
$\alpha =$	0°	1.0 00000	00001	00003	00007	00012	1	2	4	5	7
	1	00076	00092	00110	00129	00149	16	18	19	20	22
	2	00305	00336	00369	00403	00439	31	33	34	36	37
	3	00686	00732	00780	00830	00881	46	48	50	51	53
	4	01220	01282	01345	01410	01476	62	63	65	66	68
	5°	1.0 01907	01984	02063	02143	02225	77	79	80	82	83
	6	02748	02841	02935	03031	03128	93	94	96	97	99
	7	03744	03852	03962	04073	04186	1 08	10	11	13	15
	8	04896	05019	05145	05272	05400	23	26	27	28	30
	9	06204	06343	06484	06627	06771	39	41	43	44	46
	10°	1.0 07669	07824	07981	08140	08300	1 55	57	59	60	61
	11	09293	09464	09637	09812	09988	71	73	75	76	77
	12	11078	11265	11454	11645	11837	87	89	91	92	94
	13	13024	13228	13433	13640	13848	2 04	05	07	08	10
	14	15134	15354	15575	15799	16024	20	21	24	25	26
	15°	1.0 17409	17645	17884	18124	18366	2 36	39	40	42	43
	16	19851	20105	20360	20617	20876	54	55	57	59	60
	17	22463	22734	23006	23280	23556	71	72	74	76	77
	18	25247	25535	25825	26116	26409	88	90	91	93	95
	19	28205	28511	28818	29127	29438	3 06	07	09	11	13
	20°	1.0 31341	31664	31989	32316	32645	3 23	25	27	29	31
	21	34656	34997	35341	35686	36033	41	44	45	47	49
	22	38154	38514	38876	39240	39606	60	62	64	66	67
	23	41839	42218	42599	42982	43366	79	81	83	84	87
	24	45714	46112	46512	46914	47318	98	00	02	04	06
	25°	1.0 49783	50201	50620	51042	51466	4 18	19	22	24	25
	26	54049	54487	54927	55369	55812	38	40	42	43	46
	27	58518	58976	59436	59898	60363	58	60	62	65	66
	28	63193	63672	64153	64636	65122	79	81	83	86	87
$\alpha =$	29	68079	68580	69082	69587	70094	5 01	02	05	07	09
		0	1	2	3	4	0	1	2	3	4

	The function $F_0(\alpha)$						Differences				
		5	6	7	8	9	5	6	7	8	9
$\alpha =$	0°	1.0 00019	00027	00037	00049	00062	8	10	12	13	14
	1	00171	00195	00220	00247	00275	24	25	27	28	30
	2	00476	00515	00555	00597	00641	39	40	42	44	45
	3	00934	00988	01044	01101	01160	54	56	57	59	60
	4	01544	01614	01685	01757	01831	70	71	72	74	76
	5°	1.0 02308	02393	02480	02568	02657	85	87	88	89	91
	6	03227	03327	03429	03533	03638	1 00	02	04	05	06
	7	04301	04417	04534	04653	04774	16	17	19	21	22
	8	05530	05662	05795	05929	06066	32	33	34	37	38
	9	06917	07064	07213	07363	07515	47	49	50	52	54
	10°	1.0 08461	08624	08789	08956	09124	1 63	65	67	68	69
	11	10165	10345	10526	10708	10892	80	81	82	84	86
	12	12031	12226	12423	12622	12822	95	97	99	00	02
	13	14058	14270	14484	14699	14915	2 12	14	15	16	19
	14	16250	16479	16709	16940	17174	29	30	31	34	35
	15°	1.0 18609	18854	19101	19349	19599	2 45	47	48	50	52
	16	21136	21398	21662	21927	22194	62	64	65	67	69
	17	23833	24113	24394	24676	24961	80	81	82	85	86
	18	26704	27001	27299	27599	27901	97	98	00	02	04
	19	29751	30065	30381	30699	31019	3 14	16	18	20	22
	20°	1.0 32976	33308	33642	33978	34316	3 32	34	36	38	40
	21	36382	36733	37085	37440	37796	51	52	55	56	58
	22	39973	40343	40714	41087	41462	70	71	73	75	77
	23	43753	44141	44532	44924	45318	88	91	92	94	96
	24	47724	48132	48542	48954	49367	4 08	10	12	13	16
	25°	1.0 51891	52319	52748	53180	53614	4 28	29	32	34	35
	26	56258	56706	57156	57608	58062	48	50	52	54	56
	27	60829	61298	61768	62241	62716	69	70	73	75	77
	28	65609	66099	66591	67085	67581	90	92	94	96	98
$\alpha =$	29	70603	71115	71628	72144	72662	5 12	13	16	18	20
		5	6	7	8	9	5	6	7	8	9

	The function $F_0(\alpha)$						Differences				
	0	1	2	3	4		0	1	2	3	4
$\alpha = 30°$	1.0 73182	73704	74229	74756	75285		5 22	25	27	29	31
31	78507	79052	79599	80148	80700		45	47	49	52	54
32	84059	84627	85197	85770	86345		68	70	73	75	77
33	89845	90437	91031	91627	92226		92	94	96	99	0̇2
34	95871	96487	97106	97727	98351		6 16	19	21	24	26
35°	1.1 02145	02786	03430	04076	04725		6 41	44	46	49	52
36	08673	09340	10010	10682	11357		67	70	72	75	78
37	15463	16157	16853	17552	18254		94	96	99	0̇2	0̇5
38	22524	23245	23969	24696	25426		7 21	24	27	30	32
39	29864	30614	31366	32122	32880		50	52	56	58	61
40°	1.1 37493	38272	39054	39839	40627		7 79	82	85	88	91
41	45420	46230	47042	47858	48677		8 10	12	16	19	22
42	53657	54498	55342	56190	57041		41	44	48	51	54
43	62214	63088	63965	64846	65730		74	77	81	84	87
44	71105	72012	72923	73838	74757		9 07	11	15	19	22
45°	1.1 80341	81284	82231	83181	84135		9 43	47	50	54	58
46	89936	90916	91900	92887	93879		80	84	87	92	95
47	99906	0̇0925	0̇1947	0̇2973	0̇4003		10 19	22	26	30	34
48	1.2 10267	11325	12388	13454	14524		58	63	66	70	75
49	21035	22135	23239	24348	25461		11 00	04	09	13	17
50°	1.2 32229	33373	34521	35673	36830		11 44	48	52	57	62
51	43869	45058	46252	47451	48654		89	94	99	0̇3	0̇9
52	55976	57213	58455	59702	60955		12 37	42	47	53	57
53	68573	69861	71153	72451	73755		88	92	98	0̇4	0̇8
54	81685	83026	84372	85723	87080		13 41	46	51	57	63
55°	1.2 95340	96736	98139	99547	0̇0960		13 96	0̇3	0̇8	1̇3	2̇0
56	1.3 09567	11022	12483	13951	15424		14 55	61	68	73	80
57	24397	25915	27439	28969	30506		15 18	24	30	37	44
58	39867	41450	43041	44638	46242		83	91	97	0̇4	1̇1
$\alpha = 59$	56014	57667	59328	60996	62672		16 53	61	68	76	82
	0	1	2	3	4		0	1	2	3	4

	The function $F_0(\alpha)$					Differences				
	5	6	7	8	9	5	6	7	8	9
$\alpha = 30°$	1.0 75816	76350	76886	77424	77964	5 34	36	38	40	43
31	81254	81810	82369	82930	83493	56	59	61	63	66
32	86922	87502	88084	88669	89256	80	82	85	87	89
33	92828	93432	94038	94647	95258	6 04	06	09	11	13
34	98977	99605	00236	00870	01506	28	31	34	36	39
35°	1.1 05377	06031	06687	07347	08008	6 54	56	60	61	65
36	12035	12715	13398	14084	14772	80	83	86	88	91
37	18959	19666	20377	21090	21805	7 07	11	13	15	19
38	26158	26894	27632	28373	29117	36	38	41	44	47
39	33641	34406	35173	35943	36716	65	67	70	73	77
40°	1.1 41418	42213	43010	43810	44614	7 95	97	00	04	06
41	49499	50324	51153	51984	52819	8 25	29	31	35	38
42	57895	58752	59613	60477	61344	57	61	64	67	70
43	66617	67508	68402	69299	70200	91	94	97	01	05
44	75679	76604	77533	78465	79401	9 25	29	32	36	40
45°	1.1 85093	86054	87019	87988	88960	9 61	65	69	72	76
46	94874	95872	96875	97882	98892	98	03	07	10	14
47	1.2 05037	06075	07117	08163	09213	10 38	42	46	50	54
48	15599	16678	17761	18848	19939	79	83	87	91	96
49	26578	27699	28825	29955	31090	11 21	26	30	35	39
50°	1.2 37992	39158	40329	41504	42684	11 66	71	75	80	85
51	49863	51076	52293	53516	54744	12 13	17	23	28	32
52	62212	63474	64741	66013	67290	62	67	72	77	83
53	75063	76377	77696	79020	80350	13 14	19	24	30	35
54	88443	89811	91185	92564	93949	68	74	79	85	91
55°	1.3 02380	03805	05237	06674	08117	14 25	32	37	43	50
56	16904	18390	19882	21381	22886	86	92	99	05	11
57	32050	33600	35156	36720	38290	15 50	56	64	70	77
58	47853	49471	51096	52728	54367	16 18	25	32	39	47
$\alpha = 59$	64354	66045	67742	69447	71160	91	97	05	13	21
	5	6	7	8	9	5	6	7	8	9

	The function $F_0(\alpha)$					Differences				
	0	1	2	3	4	0	1	2	3	4
$\alpha = 60°$	1.3 72881	74609	76344	78088	79839	17 28	35	44	51	59
61	90513	92321	94136	95960	97792	18 08	15	24	32	41
62	1.4 08964	10856	12757	14666	16585	92	0̇1	0̇9	1̇9	2̇8
63	28288	30271	32263	34265	36276	19 83	92	0̇2	1̇1	2̇1
64	48550	50630	52721	54821	56932	20 80	91	0̇0	1̇1	2̇2
65°	1.4 69819	72005	74201	76408	78627					
66	92176	94475	96786	99108	0̇1443					
67	1.5 15709	18131	20565	23013	25474					
68	40519	43075	45644	48228	50826					
69	66721	69423	72140	74872	77620					
70°	1.5 94446	97308	0̇0187	0̇3083	0̇5996					
71	1.6 23846	26885	29942	33018	36113					
72	55097	58331	61586	64863	68160					
73	88403	91856	95332	98832	0̇2355					
74	1.7 24009	27707	31431	35181	38958					
75°	1.7 62204	66178	70183	74217	78282					
76	1.8 03335	07626	11950	16309	20703					
77	47830	52484	57177	61910	66684					
78	96216	0̇1293	0̇6415	1̇1585	1̇6803					
79	1.9 49157	54732	60363	66049	71792					
80°	2.0 07507	13680	19919	26226	32602					
81	72390	79292	86275	93343	0̇0495					
82	2.1 45325	53136	61051	69072	77202					
83	2.2 28438	37418	46533	55787	65185					
84	2.3 24844	35380	46101	57013	68123					
85°	2.4 39363	52080	65064	78326	91879					
86	2.5 80066	96060	1̇2475	2̇9333	4̇6658					
87	2.7 62073	83555	0̇5796	2̇8853	5̇2786					
88	3.0 19308	51888	86236	2̇2556	6̇1085					
$\alpha = 89$	See the adjacent table.									
	0	1	2	3	4					

For values of α which are $>65°$ one can avoid interpolation in this table by computing the desired value with the aid of the function $G_0(\alpha)$, tabulated on the next following pages.

α	$F_0(\alpha)$
89.0	3.459971
1	527003
2	601947
3	686920
4	785023

	The function $F_0(\alpha)$						Differences				
		5	6	7	8	9	5	6	7	8	9
α = 60°	1.3	81598	83365	85140	86923	88714	17	67	75	83	91	99
61		99633	01482	03339	05205	07080	18	49	57	66	75	84
62	1.4	18513	20449	22395	24350	26314	19	36	46	55	64	74
63		38297	40328	42369	44419	46479	20	31	41	50	60	71
64		59054	61186	63328	65481	67645	21	32	42	53	64	74
65	1.4	80857	83097	85350	87614	89889						
66	1.5	03790	06149	08520	10904	13300						
67		27948	30435	32935	35450	37977						
68		53438	56065	58707	61363	64034						
69		80384	83164	85960	88772	91601						
70°	1.6	08926	11874	14840	17824	20826						
71		39228	42362	45516	48689	51883						
72		71479	74819	78182	81567	84974						
73	1.7	05902	09474	13070	16691	20338						
74		42763	46595	50454	54342	58258						
75°	1.7	82378	86505	90664	94855	99079						
76	1.8	25132	29598	34100	38369	43215						
77		71499	76356	81255	86198	91185						
78	1.9	22069	27384	32750	38167	43635						
79		77593	83453	89374	95355	01400						
80°	2.0	39049	45567	52159	58825	65569						
81	2.1	07736	15065	22487	30003	37614						
82		85444	93800	02275	10870	19590						
83	2.2	74730	84428	94284	04301	14486						
84	2.3	79438	90965	02713	14689	26903						
85°	2.5	05734	19907	34410	49259	64472						
86	2.6	64477	82819	01713	21195	41301						
87	2.8	77664	03564	30572	58788	88324						
88	3.2	02109	45972	93094	43998	09341						
α = 89	See the adjacent table.											
		5	6	7	8	9						

For values of α which are $>65°$ one can avoid interpolation in this table by computing the desired value with the aid of the function $G_0(\alpha)$, tabulated on the next following pages.

α	$F_0(\alpha)$
89.5	3.901065
6	4.043099
7	226224
8	484337
9	925599

$$F_0(\alpha) = G_0(\alpha) - a \cdot {}^{10}\log(90 - \alpha^\circ),\ a = \frac{2}{\pi} \cdot {}^{e}\log 10 = 1.465871$$

Ex. $\alpha = 83.2543^\circ$, $90 - \alpha = 6.7457$, ${}^{10}\log = 0.829027$,
$G_0 = 3.466787$, $F_0 = G_0 - a \cdot 0.829027 = 2.251540$.

	The function $G_0(\alpha)$						Differences				
		0	1	2	3	4	0	1	2	3	4
$\alpha = 65^\circ$	3.5	19019	18653	18288	17923	17559	3 66	65	65	64	64
66		15388	15029	14670	14312	13955	59	59	58	57	57
67		11827	11474	11123	10772	10422	53	51	51	50	49
68		08338	07993	07649	07306	06963	45	44	43	43	42
69		04924	04587	04251	03915	03580	37	36	36	35	34
70°	3.5	01588	01259	00931	00604	00277	3 29	28	27	27	26
71	3.4	98334	98013	97693	97374	97056	21	20	19	18	18
72		95164	94852	94541	94231	93921	12	11	10	10	08
73		92083	91780	91478	91177	90876	03	02	01	01	$\bar{9}9$
74		89094	88801	88508	88216	87925	2 93	93	92	91	89
75°	3.4	86202	85918	85636	85354	85073	2 84	82	82	81	79
76		83411	83138	82866	82595	82325	73	72	71	70	69
77		80728	80465	80204	79944	79685	63	61	60	59	57
78		78157	77906	77656	77408	77161	51	50	48	47	46
79		75705	75466	75229	74993	74759	39	37	36	34	34
80°	3.4	73379	73153	72929	72707	72485	2 26	24	22	22	20
81		71187	70975	70765	70557	70350	12	10	08	07	06
82		69138	68942	68747	68553	68361	1 96	95	94	92	90
83		67243	67062	66883	66706	66531	81	79	77	75	74
84		65513	65350	65188	65029	64871	63	62	59	58	56
85°	3.4	63963	63818	63676	63535	63396	1 45	42	41	39	36
86		62608	62485	62363	62243	62126	23	22	20	17	15
87		61471	61370	61272	61176	61083	01	$\bar{9}8$	$\bar{9}6$	$\bar{9}3$	$\bar{9}1$
88		60579	60504	60433	60364	60299	0 75	71	69	65	63
$\alpha = 89$		59971	59928	59889	59853	59822	43	39	36	31	28
		0	1	2	3	4	0	1	2	3	4

$$F_0(\alpha) = G_0(\alpha) - a \cdot {}^{10}\log(90 - \alpha^\circ),\ a = \frac{2}{\pi} \cdot {}^{\circ}\log 10 = 1.465871$$

Ex. $\alpha = 89.5379^\circ$, $90 - \alpha = 0.4621$, ${}^{10}\log = 0.664736 - 1$,
$G_0 = 3.459785$, $F_0 = G_0 + a \cdot 0.335264 = 3.951248$.

	The function $G_0(\alpha)$						Differences				
		5	6	7	8	9	5	6	7	8	9
$\alpha =$ 65°	3.5	17195	16832	16470	16109	15748	3 63	62	61	61	60
66		13598	13243	12888	12533	12180	55	55	55	53	53
67		10073	09724	09377	09030	08683	49	47	47	47	45
68		06621	06280	05940	05601	05262	41	40	39	39	38
69		03246	02913	02581	02249	01918	33	32	32	31	30
70°	3.4	99951	99626	99302	98978	98656	3 25	24	24	22	22
71		96738	96422	96106	95791	95477	16	16	15	14	13
72		93613	93305	92998	92692	92387	08	07	06	05	04
73		90577	90278	89981	89684	89389	2 99	97	97	95	95
74		87636	87347	87059	86772	86487	89	88	87	85	85
75°	3.4	84794	84515	84238	83961	83686	2 79	77	77	75	75
76		82056	81788	81521	81256	80991	68	67	65	65	63
77		79428	79171	78916	78661	78408	57	55	55	53	51
78		76915	76671	76427	76185	75944	44	44	42	41	39
79		74525	74293	74063	73833	73605	32	30	30	28	26
80°	3.4	72265	72047	71830	71614	71400	2 18	17	16	14	13
81		70144	69940	69737	69536	69336	04	03	01	00	$\bar{98}$
82		68171	67982	67795	67609	67425	1 89	87	86	84	82
83		66357	66184	66014	65845	65678	73	70	69	67	65
84		64715	64560	64408	64258	64109	55	52	50	49	46
85°	3.4	63260	63125	62993	62863	62734	1 35	32	30	29	26
86		62011	61898	61788	61680	61574	13	10	08	06	03
87		60992	60904	60819	60736	60656	0 88	85	83	80	77
88		60236	60177	60120	60067	60017	59	57	53	50	46
$\alpha =$ 89		59794	59770	59751	59737	59727	24	19	14	10	03
		5	6	7	8	9	5	6	7	8	9

	The function $E_0(\alpha)$						Differences				
		0	1	2	3	4	0	1	2	3	4
$\alpha =$	0°	1.0 00000	$\bar{9}9999$	$\bar{9}9997$	$\bar{9}9993$	$\bar{9}9988$	1	2	4	5	7
	1	0.9 99924	99908	99890	99871	99851	16	18	19	20	22
	2	99695	99664	99631	99597	99561	31	33	34	36	37
	3	99315	99269	99221	99171	99120	46	48	50	51	52
	4	98782	98721	98658	98593	98527	61	63	65	66	68
	5°	0.9 98098	98022	97943	97863	97782	76	79	80	81	83
	6	97263	97171	97078	96983	96887	92	93	95	96	98
	7	96277	96170	96061	95951	95840	1 07	09	10	11	13
	8	95140	95018	94895	94770	94643	22	23	25	27	28
	9	93854	93717	93579	93439	93297	37	38	40	42	43
	10°	0.9 92418	92267	92114	91959	91803	1 51	53	55	56	58
	11	90835	90668	90500	90331	90160	67	68	69	71	72
	12	89104	88923	88740	88556	88371	81	83	84	85	87
	13	87227	87031	86834	86635	86435	96	97	99	$\dot{0}0$	$\dot{0}2$
	14	85204	84994	84782	84569	84354	2 10	12	13	15	16
	15°	0.9 83037	82812	82586	82359	82130	2 25	26	27	29	30
	16	80727	80488	80248	80006	79763	39	40	42	43	45
	17	78274	78021	77767	77511	77254	53	54	56	57	58
	18	75682	75415	75147	74877	74606	67	68	70	71	73
	19	72950	72669	72387	72104	71819	81	82	83	85	86
	20°	0.9 70081	69786	69490	69193	68895	2 95	96	97	98	$\dot{0}0$
	21	67075	66767	66458	66148	65836	3 08	09	10	12	14
	22	63936	63615	63292	62968	62643	21	23	24	25	27
	23	60664	60329	59994	59657	59318	35	35	37	39	39
	24	57261	56914	56565	56215	55864	47	49	50	51	53
	25°	0.9 53730	53369	53008	52645	52281	3 61	61	63	64	65
	26	50071	49699	49325	48950	48573	72	74	75	77	78
	27	46289	45904	45517	45130	44741	85	87	87	89	90
	28	42384	41986	41588	41189	40788	98	98	99	$\dot{0}1$	$\dot{0}2$
$\alpha =$	29	38358	37949	37539	37128	36715	4 09	10	11	13	14
		0	1	2	3	4	0	1	2	3	4

		The function $E_0(\alpha)$					Differences				
		5	6	7	8	9	5	6	7	8	9
$\alpha =$	0°	0.9 99981	99973	99963	99951	99938	8	10	12	13	14
	1	99829	99805	99780	99753	99725	24	25	27	28	30
	2	99524	99485	99445	99403	99360	39	40	42	43	45
	3	99068	99014	98958	98901	98842	54	56	57	59	60
	4	98459	98390	98319	98247	98173	69	71	72	74	75
	5°	0.9 97699	97615	97529	97442	97353	84	86	87	89	90
	6	96789	96689	96588	96486	96382	1 00	01	02	04	05
	7	95727	95613	95497	95379	95260	14	16	18	19	20
	8	94515	94386	94255	94123	93989	29	31	32	34	35
	9	93154	93010	92864	92717	92568	44	46	47	49	50
	10°	0.9 91645	91486	91325	91163	91000	1 59	61	62	63	65
	11	89988	89814	89639	89462	89284	74	75	77	78	80
	12	88184	87995	87805	87614	87421	89	90	91	93	94
	13	86233	86030	85826	85620	85413	2 03	04	06	07	09
	14	84138	83921	83702	83482	83260	17	19	20	22	23
	15°	0.9 81900	81668	81435	81200	80964	2 32	33	35	36	37
	16	79518	79272	79025	78776	78526	46	47	49	50	52
	17	76996	76736	76474	76211	75947	60	62	63	64	65
	18	74333	74059	73784	73507	73229	74	75	77	78	79
	19	71533	71245	70956	70666	70374	88	89	90	92	93
	20°	0.9 68595	68294	67991	67687	67382	3 01	03	04	05	07
	21	65522	65208	64892	64574	64256	14	16	18	18	20
	22	62316	61988	61659	61329	60997	28	29	30	32	33
	23	58979	58638	58295	57952	57607	41	43	43	45	46
	24	55511	55157	54802	54446	54088	54	55	56	58	58
	25°	0.9 51916	51550	51182	50813	50443	3 66	68	69	70	72
	26	48195	47817	47436	47055	46673	78	81	81	82	84
	27	44351	43960	43568	43174	42780	91	92	94	94	96
	28	40386	39983	39578	39173	38766	4 03	05	05	07	08
$\alpha =$	29	36301	35887	35471	35053	34635	14	16	18	18	20
		5	6	7	8	9	5	6	7	8	9

	The function $E_0(\alpha)$						Differences				
		0	1	2	3	4	0	1	2	3	4
$\alpha =$ 30°	0.9	34215	33795	33373	32950	32526	4 20	22	23	24	25
31		29957	29525	29092	28658	28222	32	33	34	36	36
32		25586	25143	24699	24254	23807	43	44	45	47	47
33		21106	20652	20196	19740	19283	54	56	56	57	58
34		16517	16053	15587	15120	14653	64	66	67	67	69
35°	0.9	11825	11350	10874	10397	09919	4 75	76	77	78	79
36		07031	06546	06060	05573	05085	85	86	87	88	89
37		02138	01643	01148	00651	00154	95	95	97	97	98
38	0.8	97150	96646	96141	95635	95129	5 04	05	06	06	08
39		92070	91557	91043	90528	90013	13	14	15	15	17
40°	0.8	86901	86379	85856	85333	84809	5 22	23	23	24	25
41		81646	81116	80585	80054	79521	30	31	31	33	33
42		76310	75772	75233	74694	74153	38	39	39	41	41
43		70895	70350	69803	69256	68708	45	47	47	48	48
44		65406	64853	64300	63746	63191	53	53	54	55	56
45°	0.8	59847	59287	58727	58166	57604	5 60	60	61	62	62
46		54220	53654	53087	52520	51952	66	67	67	68	69
47		48531	47959	47386	46813	46239	72	73	73	74	74
48		42784	42206	41628	41049	40470	78	78	79	79	80
49		36983	36400	35817	35233	34648	83	83	84	85	85
50°	0.8	31132	30544	29956	29368	28779	5 88	88	88	89	89
51		25236	24644	24052	23459	22866	92	92	93	93	93
52		19300	18704	18108	17512	16915	96	96	96	97	97
53		13329	12730	12131	11531	10931	99	99	00	00	00
54		07327	06726	06124	05522	04919	6 01	02	02	03	02
55°	0.8	01300	00696	00092	99488	98884	6 04	04	04	04	05
56	0.7	95254	94648	94043	93437	92831	06	05	06	06	06
57		89193	88586	87979	87373	86766	07	07	06	07	07
58		83123	82516	81909	81302	80694	07	07	07	08	07
$\alpha =$ 59		77051	76444	75837	75230	74623	07	07	07	07	07
		0	1	2	3	4	0	1	2	3	4

	The function $E_0(\alpha)$						Differences				
		5	6	7	8	9	5	6	7	8	9
$\alpha =$ 30°	0.9	32101	31674	31247	30818	30388	4 27	27	29	30	31
31		27786	27348	26909	26470	26029	38	39	39	41	43
32		23360	22911	22461	22010	21559	49	50	51	51	53
33		18825	18365	17905	17443	16981	60	60	62	62	64
34		14184	13714	13243	12771	12299	70	71	72	72	74
35°	0.9	09440	08960	08479	07997	07514	4 80	81	82	83	83
36		04596	04107	03616	03124	02632	89	91	92	92	94
37	0.8	99656	99156	98656	98155	97653	5 00	00	01	02	03
38		94621	94113	93603	93093	92582	08	10	10	11	12
39		89496	88979	88460	87941	87421	17	19	19	20	20
40°	0.8	84284	83758	83231	82704	82175	5 26	27	27	29	29
41		78988	78454	77919	77384	76847	34	35	35	37	37
42		73612	73070	72528	71984	71440	42	42	44	44	45
43		68160	67611	67061	66510	65958	49	50	51	52	52
44		62635	62079	61522	60964	60406	56	57	58	58	59
45°	0.8	57042	56479	55915	55351	54786	5 63	64	64	65	66
46		51383	50814	50244	49674	49103	69	70	70	71	72
47		45665	45090	44514	43938	43361	75	76	76	77	77
48		39890	39310	38729	38147	37565	80	81	82	82	82
49		34063	33478	32892	32306	31719	85	86	86	87	87
50°	0.8	28190	27600	27010	26419	25828	5 90	90	91	91	92
51		22273	21679	21085	20490	19895	94	94	95	95	95
52		16318	15721	15123	14525	13927	97	98	98	98	98
53		10331	09731	09131	08530	07929	6 00	00	01	01	02
54		04317	03714	03111	02507	01904	03	03	04	03	04
55°	0.7	98279	97674	97069	96464	95859	6 05	05	05	05	05
56		92225	91619	91012	90406	89799	06	07	06	07	06
57		86159	85552	84945	84338	83730	07	07	07	08	07
58		80087	79480	78873	78265	77658	07	07	08	07	07
$\alpha =$ 59		74016	73409	72802	72195	71589	07	07	07	06	07
		5	6	7	8	9	5	6	7	8	9

	The function $E_0(\alpha)$						Differences				
		0	1	2	3	4	0	1	2	3	4
$\alpha =$ 60°	0.7	70982	70376	69769	69163	68557	6 06	07	06	06	06
61		64923	64318	63713	63108	62503	05	05	05	05	04
62		58880	58277	57674	57071	56469	03	03	03	02	03
63		52860	52259	51659	51059	50459	01	00	00	00	$\bar{9}9$
64		46869	46272	45675	45079	44483	5 97	97	96	96	96
65°	0.7	40916	40323	39730	39138	38547	5 93	93	92	91	91
66		35007	34419	33832	33245	32658	88	87	87	87	86
67		29151	28569	27987	27406	26825	82	82	81	81	80
68		23356	22780	22205	21630	21056	76	75	75	74	73
69		17629	17061	16493	15926	15360	68	68	67	66	65
70°	0.7	11981	11421	10862	10304	09746	5 60	59	58	58	56
71		06421	05870	05320	04771	04223	51	50	49	48	46
72		00958	00417	$\bar{9}9878$	$\bar{9}9340$	$\bar{9}8802$	41	39	38	38	36
73	0.6	95603	95074	94546	94019	93493	29	28	27	26	24
74		90367	89850	89335	88821	88308	17	15	14	13	11
75°	0.6	85261	84758	84256	83756	83258	5 03	02	00	$\bar{9}8$	$\bar{9}7$
76		80298	79810	79324	78839	78356	4 88	86	85	83	82
77		75491	75020	74550	74082	73616	71	70	68	66	65
78		70855	70402	69950	69500	69052	53	52	50	48	46
79		66405	65971	65539	65109	64681	34	32	30	28	26
80°	0.6	62157	61744	61334	60925	60519	4 13	10	09	06	04
81		58131	57741	57354	56969	56587	3 90	87	85	82	80
82		54346	53981	53619	53260	52904	65	62	59	56	54
83		50824	50488	50154	49823	49496	36	34	31	27	25
84		47593	47287	46984	46685	46388	06	03	$\bar{9}9$	$\bar{9}7$	$\bar{9}3$
85°	0.6	44682	44409	44141	43876	43614	2 73	68	65	62	57
86		42125	41891	41660	41434	41212	34	31	26	22	19
87		39967	39776	39588	39406	39228	1 91	88	82	78	73
88		38265	38122	37985	37854	37728	43	37	31	26	20
$\alpha =$ 89		37098	37016	36940	36871	36810	0 82	76	69	61	54
		0	1	2	3	4	0	1	2	3	4

	The function $E_0(\alpha)$					Differences				
	→ 5	6	7	8	9	5	6	7	8	9
$\alpha = 60°$	0.7 67951	67345	66739	66134	65528	6 06	06	05	06	05
61	61899	61295	60691	60087	59483	04	04	04	04	03
62	55866	55265	54663	54061	53460	01	02	02	01	00
63	49860	49261	48663	48065	47467	5 99	98	98	98	98
64	43887	43292	42697	42103	41509	95	95	94	94	93
65°	0.7 37956	37365	36775	36185	35596	5 91	90	90	89	89
66	32072	31487	30902	30318	29734	85	85	84	84	83
67	26245	25666	25087	24509	23932	79	79	78	77	76
68	20483	19911	19339	18769	18199	72	72	70	70	70
69	14795	14231	13667	13104	12542	64	64	63	62	61
70°	0.7 09190	08634	08079	07526	06973	5 56	55	53	53	52
71	03677	03131	02586	02042	01500	46	45	44	42	42
72	0.6 98266	97731	97197	96665	96133	35	34	32	32	30
73	92969	92446	91924	91404	90884	23	22	20	20	17
74	87797	87287	86778	86271	85765	10	09	07	06	04
75°	0.6 82761	82265	81771	81278	80787	4 96	94	93	91	89
76	77874	77394	76916	76440	75965	80	78	76	75	74
77	73151	72688	72227	71768	71311	63	61	59	57	56
78	68606	68162	67720	67279	66841	44	42	41	38	36
79	64255	63831	63409	62990	62573	24	22	19	17	16
80°	0.6 60115	59714	59314	58917	58523	4 01	00	$\bar{9}7$	$\bar{9}4$	$\bar{9}2$
81	56207	55829	55454	55082	54713	3 78	75	72	69	67
82	52550	52200	51852	51506	51164	50	48	46	42	40
83	49171	48849	48530	48215	47902	22	19	15	13	09
84	46095	45806	45520	45237	44958	2 89	86	83	79	76
85°	0.6 43357	43103	42852	42606	42364	2 54	51	46	42	39
86	40993	40780	40570	40365	40164	13	10	05	01	$\bar{9}7$
87	39055	38887	38724	38566	38413	1 68	63	58	53	48
88	37608	37494	37385	37283	37188	14	09	02	$\bar{9}5$	$\bar{9}0$
$\alpha = 89$	36756	36711	36673	36645	36627	0 45	38	28	18	07
	→ 5	6	7	8	9	5	6	7	8	9

The function $A_0(\alpha, \beta)$

	$\alpha = 0°$		1°	2°	3°	4°	5°	6°	7°
$\beta =$ 0°	0.0	00000	00000	00000	00000	00000	00000	00000	00000
1		17452	17451	17447	17440	17431	17419	17405	17387
2		34899	34897	34889	34876	34857	34833	34804	34770
3		52336	52332	52320	52300	52272	52236	52193	52141
4		69756	69751	69735	69709	69672	69624	69566	69497
5°	0.0	87156	87149	87129	87096	87050	86990	86917	86831
6	0.1	04528	04521	04497	04457	04401	04330	04242	04139
7		21869	21860	21832	21786	21721	21638	21536	21416
8		39173	39163	39131	39078	39004	38908	38792	38655
9		56434	56423	56387	56327	56244	56137	56006	55852
10°	0.1	73648	73635	73595	73529	73437	73318	73173	73002
11		90809	90794	90751	90678	90577	90446	90287	90099
12	0.2	07912	07896	07848	07769	07659	07516	07343	07138
13		24951	24934	24883	24797	24677	24523	24335	24114
14		41922	41903	41848	41756	41627	41462	41260	41021
15°	0.2	58819	58799	58740	58642	58504	58327	58111	57856
16		75637	75616	75553	75449	75302	75113	74883	74611
17		92372	92349	92283	92171	92016	91816	91572	91283
18	0.3	09017	08993	08923	08805	08641	08429	08171	07867
19		25568	25543	25469	25345	25172	24949	24677	24356
20°	0.3	42020	41994	41916	41786	41604	41370	41084	40747
21		58368	58341	58259	58122	57932	57687	57387	57034
22		74607	74578	74493	74350	74151	73894	73582	73213
23		90731	90701	90612	90463	90255	89988	89662	89277
24	0.4	06737	06706	06613	06458	06242	05963	05624	05223
25°	0.4	22618	22586	22490	22329	22104	21815	21462	21046
26		38371	38338	38238	38071	37838	37538	37172	36740
27		53990	53956	53852	53680	53438	53128	52749	52302
28		69472	69436	69329	69150	68900	68579	68188	67725
29		84810	84773	84662	84478	84220	83888	83484	83007
$\beta =$ 30°	0.5	00000	$\bar{9}9962$	$\bar{9}9848$	$\bar{9}9658$	$\bar{9}9391$	$\bar{9}9050$	$\bar{9}8633$	$\bar{9}8141$
	$\alpha = 0°$		1°	2°	3°	4°	5°	6°	7°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 8°$	9°	10°	11°	12°	13°	14°	15°
$\beta =$ 0°	0.0 00000	00000	00000	00000	00000	00000	00000	00000
1	17368	17345	17320	17292	17262	17229	17194	17156
2	34730	34685	34635	34580	34519	34454	34383	34307
3	52082	52014	51939	51856	51766	51667	51562	51448
4	69417	69328	69228	69117	68996	68866	68724	68573
5°	0.0 86732	86620	86495	86357	86206	86043	85866	85677
6	0.1 04020	03886	03736	03571	03390	03193	02982	02756
7	21277	21120	20945	20753	20542	20313	20067	19803
8	38497	38318	38118	37898	37657	37396	37114	36813
9	55674	55473	55249	55001	54730	54437	54121	53782
10°	0.1 72804	72581	72332	72057	71757	71431	71080	70704
11	89882	89636	89363	89061	88731	88373	87987	87574
12	0.2 06901	06634	06336	06007	05647	05257	04837	04388
13	23858	23569	23246	22890	22501	22080	21625	21139
14	40746	40436	40089	39706	39288	38834	38346	37822
15°	0.2 57562	57229	56858	56448	56001	55516	54994	54434
16	74298	73944	73549	73113	72637	72120	71564	70968
17	90951	90576	90157	89694	89189	88642	88052	87420
18	0.3 07516	07119	06676	06188	05654	05075	04452	03784
19	23987	23569	23102	22588	22025	21416	20759	20056
20°	0.3 40359	39920	39430	38889	38299	37659	36969	36231
21	56628	56167	55654	55088	54469	53799	53077	52304
22	72787	72306	71770	71178	70532	69831	69077	68269
23	88834	88332	87773	87156	86482	85752	84965	84123
24	0.4 04762	04240	03658	03016	02314	01554	00736	99860
25°	0.4 20566	20024	19419	18753	18024	17235	16385	15475
26	36243	35681	35054	34362	33607	32788	31907	30964
27	51787	51205	50555	49839	49058	48210	47299	46323
28	67193	66591	65920	65180	64372	63496	62554	61545
29	82457	81835	81142	80379	79545	78641	77668	76628
$\beta =$ 30°	0.4 97574	96933	96219	95431	94572	93640	92638	91565
	$\alpha = 8°$	9°	10°	11°	12°	13°	14°	15°

The function $\Lambda_0(\alpha, \beta)$

		$\alpha = 0°$	1°	2°	3°	4°	5°	6°	7°
$\beta = 30°$	0.5	00000	$\bar{9}9962$	$\bar{9}9848$	$\bar{9}9658$	$\bar{9}9391$	$\bar{9}9050$	$\bar{9}8633$	$\bar{9}8141$
31		15038	14999	14881	14685	14411	14059	13630	13123
32		29919	29879	29758	29556	29274	28912	28470	27949
33		44639	44598	44473	44266	43976	43604	43150	42614
34		59193	59150	59023	58810	58512	58130	57664	57114
35°	0.5	73576	73533	73402	73184	72878	72487	72009	71445
36		87785	87740	87606	87383	87070	86669	86179	85601
37	0.6	01815	01769	01632	01403	01083	00672	00170	$\bar{9}9579$
38		15661	15615	15474	15240	14912	14492	13979	13374
39		29320	29272	29129	28889	28555	28125	27601	26983
40°	0.6	42788	42739	42592	42347	42006	41567	41032	40400
41		56059	56009	55859	55610	55261	54813	54267	53623
42		69131	69080	68927	68672	68317	67860	67303	66647
43		81998	81946	81791	81531	81169	80703	80136	79467
44		94658	94605	94447	94183	93814	93340	92762	92081
45°	0.7	07107	07053	06891	06623	06247	05765	05177	04484
46		19340	19285	19121	18847	18465	17975	17377	16672
47		31354	31298	31131	30853	30464	29966	29358	28642
48		43145	43088	42919	42636	42241	41735	41118	40390
49		54710	54652	54480	54193	53792	53278	52651	51913
50°	0.7	66044	65986	65811	65520	65113	64592	63956	63207
51		77146	77087	76909	76614	76201	75672	75028	74269
52		88011	87951	87771	87471	87053	86517	85863	85094
53		98636	98575	98392	98089	97665	97122	96460	95681
54	0.8	09017	08955	08771	08463	08034	07484	06814	06026
55°	0.8	19152	19090	18903	18592	18157	17600	16922	16125
56		29038	28974	28785	28470	28031	27467	26782	25976
57		38671	38607	38415	38097	37652	37083	36390	35575
58		48048	47984	47790	47468	47019	46443	45743	44920
59		57167	57102	56906	56581	56127	55546	54839	54008
$\beta = 60°$	0.8	66025	65959	65762	65433	64975	64388	63674	62836
		$\alpha = 0°$	1°	2°	3°	4°	5°	6°	7°

The function $\Lambda_0(\alpha, \beta)$

		$\alpha = 8°$	9°	10°	11°	12°	13°	14°	15°
$\beta = 30°$	0.4	97574	96933	96219	95431	94572	93640	92638	91565
31	0.5	12539	11879	11144	10333	09448	08489	07457	06354
32		27349	26670	25913	25080	24169	23184	22123	20988
33		41997	41300	40523	39666	38732	37719	36630	35465
34		56481	55765	54968	54089	53130	52091	50974	49779
35°	0.5	70795	70062	69244	68343	67360	66295	65150	63926
36		84936	84184	83347	82424	81417	80327	79155	77901
37		98898	98129	97272	96328	95297	94182	92984	91702
38	0.6	12678	11891	11015	10050	08997	07857	06632	05323
39		26272	25468	24573	23587	22511	21348	20097	18760
40°	0.6	39674	38854	37940	36934	35836	34649	33373	32010
41		52882	52045	51113	50087	48968	47758	46457	45068
42		65891	65038	64088	63043	61903	60670	59345	57931
43		78698	77829	76861	75797	74636	73381	72033	70594
44		91297	90413	89428	88345	87164	85888	84517	83054
45°	0.7	03687	02787	01786	00684	$\bar{9}9484$	$\bar{9}8186$	$\bar{9}6794$	$\bar{9}5307$
46		15862	14947	13930	12810	11591	10273	08859	07350
47		27819	26890	25856	24720	23482	22144	20709	19179
48		39554	38611	37562	36409	35153	33796	32341	30790
49		51065	50108	49044	47874	46601	45226	43752	42180
50°	0.7	62347	61376	60298	59112	57822	56430	54937	53346
51		73397	72413	71320	70120	68814	67404	65894	64285
52		84211	83215	82109	80893	79572	78146	76619	74993
53		94787	93779	92659	91430	90094	88653	87110	85467
54	0.8	05121	04101	02969	01726	00376	$\bar{9}8920$	$\bar{9}7362$	$\bar{9}5705$
55°	0.8	15210	14179	13034	11779	10416	08946	07375	05703
56		25050	24009	22853	21586	20210	18728	17143	15459
57		34640	33589	32422	31144	29756	28262	26666	24970
58		43977	42915	41738	40449	39051	37547	35940	34234
59		53056	51985	50799	49500	48092	46578	44962	43248
$\beta = 60°$	0.8	61876	60797	59602	58294	56877	55355	53731	52010
		$\alpha = 8°$	9°	10°	11°	12°	13°	14°	15°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 0°$		1°	2°	3°	4°	5°	6°	7°
$\beta = 60°$	0.8	66025	65959	65762	65433	64975	64388	63674	62836
61		74620	74553	74354	74022	73559	72967	72247	71402
62		82948	82880	82679	82344	81877	81280	80554	79702
63		91007	90939	90735	90398	89927	89324	88593	87735
64		98794	98726	98521	98180	97705	97098	96362	95498
65°	0.9	06308	06239	06032	05689	05210	04599	03857	02989
66		13545	13476	13268	12922	12440	11824	11078	10205
67		20505	20435	20225	19877	19392	18772	18022	17145
68		27184	27113	26902	26551	26063	25441	24687	23807
69		33580	33509	33297	32944	32453	31827	31071	30188
70°	0.9	39693	39621	39407	39052	38559	37930	37172	36287
71		45519	45447	45231	44875	44379	43748	42988	42102
72		51057	50984	50768	50409	49911	49279	48517	47632
73		56305	56232	56014	55654	55155	54522	53760	52875
74		61262	61189	60970	60608	60108	59474	58713	57830
75°	0.9	65926	65852	65633	65270	64769	64135	63376	62498
76		70296	70222	70002	69638	69137	68504	67747	66876
77		74370	74296	74075	73711	73210	72579	71827	70965
78		78148	78073	77852	77487	76987	76360	75615	74764
79		81627	81553	81331	80966	80468	79846	79111	78275
80°	0.9	84808	84733	84511	84147	83652	83037	82315	81498
81		87688	87613	87391	87029	86539	85934	85229	84437
82		90268	90193	89971	89612	89129	88538	87855	87095
83		92546	92471	92250	91894	91422	90851	90197	89478
84		94522	94447	94227	93878	93421	92877	92263	91597
85°	0.9	96195	96120	95903	95564	95130	94624	94063	93463
86		97564	97489	97277	96957	96557	96104	95613	95096
87		98630	98555	98353	98062	97717	97338	96938	96524
88		99391	99319	99139	98902	98638	98361	98075	97784
89		99848	99785	99660	99519	99372	99224	99074	98924
$\beta = 90°$	1.0	00000	00000	00000	00000	00000	00000	00000	00000
	$\alpha = 0°$		1°	2°	3°	4°	5°	6°	7°

The function $\Lambda_0(\alpha, \beta)$

		$\alpha = 8°$	9°	10°	11°	12°	13°	14°	15°
$\beta = 60°$	0.8	61876	60797	59602	58294	56877	55355	53731	52010
61		70434	69347	68144	66829	65404	63875	62244	60517
62		78728	77634	76423	75101	73670	72135	70500	68769
63		86754	85654	84437	83109	81673	80133	78495	76762
64		94511	93405	92183	90850	89410	87868	86229	84496
65°	0.9	01997	00886	$\bar{9}9660$	$\bar{9}8323$	$\bar{9}6881$	$\bar{9}5338$	$\bar{9}3699$	$\bar{9}1969$
66		09209	08094	06865	05526	04083	02541	00905	$\bar{9}9179$
67		16145	15028	13796	12457	11015	09476	07844	06126
68		22804	21684	20453	19114	17675	16141	14518	12810
69		29183	28063	26832	25497	24063	22537	20923	19229
70°	0.9	35282	34162	32934	31604	30177	28661	27061	25384
71		41097	39980	38756	37433	36017	34515	32932	31275
72		46629	45516	44299	42986	41583	40097	38535	36903
73		51876	50769	49561	48261	46875	45410	43872	42269
74		56837	55738	54543	53258	51893	50453	48945	47376
75°	0.9	61512	60424	59244	57980	56638	55228	53755	52226
76		65900	64826	63666	62425	61114	59739	58306	56823
77		70001	68946	67809	66598	65322	63988	62602	61171
78		73817	72785	71676	70500	69266	67979	66647	65276
79		77349	76344	75271	74137	72951	71720	70449	69145
80°	0.9	80599	79628	78597	77513	76384	75216	74016	72787
81		83571	82642	81661	80635	79572	78477	77356	76212
82		86271	85393	84472	83515	82527	81515	80482	79432
83		88706	87890	87040	86162	85262	84342	83408	82461
84		90888	90147	89380	88594	87791	86976	86150	85316
85°	0.9	92833	92180	91511	90828	90135	89434	88727	88015
86		94561	94012	93454	92888	92317	91741	91162	90580
87		96102	95672	95239	94802	94362	93921	93479	93035
88		97491	97195	96898	96601	96302	96003	95704	95405
89		98773	98622	98471	98320	98169	98019	97868	97717
$\beta = 90°$	1.0	00000	00000	00000	00000	00000	00000	00000	00000
		$\alpha = 8°$	9°	10°	11°	12°	13°	14°	15°

The function $A_0(\alpha, \beta)$

	α = 15°	16°	17°	18°	19°	20°	21°	22°
β = 0°	0.0 00000	00000	00000	00000	00000	00000	00000	00000
1	17156	17116	17073	17028	16980	16930	16878	16823
2	34307	34227	34141	34051	33955	33855	33750	33641
3	51448	51327	51199	51063	50920	50770	50613	50449
4	68573	68412	68241	68060	67870	67670	67460	67241
5°	0.0 85677	85476	85263	85037	84799	84549	84287	84013
6	0.1 02756	02514	02258	01987	01702	01402	01088	00760
7	19803	19521	19223	18907	18574	18225	17859	17477
8	36813	36492	36151	35790	35410	35012	34594	34157
9	53782	53421	53038	52633	52206	51758	51288	50798
10°	0.1 70704	70303	69878	69429	68955	68458	67937	67393
11	87574	87134	86667	86173	85653	85106	84534	83937
12	0.2 04388	03908	03399	02861	02294	01699	01076	00425
13	21139	20620	20069	19488	18875	18231	17557	16854
14	37822	37265	36673	36048	35389	34697	33973	33216
15°	0.2 54434	53838	53205	52536	51831	51092	50317	49509
16	70968	70333	69660	68948	68198	67410	66586	65726
17	87420	86747	86032	85278	84483	83648	82774	81862
18	0.3 03784	03073	02319	01521	00682	$\bar{9}9800$	$\bar{9}8877$	$\bar{9}7914$
19	20056	19307	18513	17673	16789	15861	14890	13876
20°	0.3 36231	35445	34611	33729	32801	31827	30807	29743
21	52304	51480	50607	49684	48712	47692	46625	45510
22	68269	67409	66496	65532	64517	63452	62337	61174
23	84123	83226	82275	81270	80212	79102	77941	76729
24	99860	98927	97937	96892	95792	94638	93430	92170
25°	0.4 15475	14506	13479	12394	11252	10054	08800	07492
26	30964	29960	28895	27771	26588	25346	24047	22693
27	46323	45283	44182	43018	41794	40510	39167	37766
28	61545	60472	59333	58131	56867	55540	54153	52707
29	76628	75520	74345	73106	71801	70433	69003	67512
β = 30°	0.4 91565	90424	89214	87937	86593	85184	83712	82176
	α = 15°	16°	17°	18°	19°	20°	21°	22°

The function $\Lambda_0(\alpha, \beta)$

	α = 23°	24°	25°	26°	27°	28°	29°	30°
β = 0°	0.0 00000	00000	00000	00000	00000	00000	00000	00000
1	16766	16707	16645	16581	16515	16447	16377	16304
2	33527	33408	33285	33157	33025	32889	32748	32604
3	50277	50099	49915	49723	49525	49321	49110	48894
4	67013	66776	66530	66274	66011	65738	65458	65169
5°	0.0 83728	83432	83124	82806	82476	82136	81786	81425
6	0.1 00418	00063	$\bar{9}$9694	$\bar{9}$9312	$\bar{9}$8917	$\bar{9}$8510	$\bar{9}$8089	$\bar{9}$7657
7	17078	16664	16234	15789	15329	14854	14364	13860
8	33703	33230	32739	32231	31706	31163	30604	30029
9	50287	49756	49204	48633	48043	47434	46806	46160
10°	0.1 66826	66236	65625	64991	64336	63661	62964	62247
11	83314	82667	81995	81300	80581	79838	79074	78287
12	99747	99042	98311	97554	96771	95963	95130	94273
13	0.2 16120	15358	14567	13749	12902	12029	11129	10203
14	32428	31609	30759	29879	28970	28031	27064	26070
15°	0.2 48666	47790	46882	45941	44969	43966	42933	41870
16	64829	63897	62931	61930	60896	59829	58729	57599
17	80912	79925	78900	77840	76744	75613	74449	73251
18	96911	95868	94786	93667	92510	91316	90087	88823
19	0.3 12820	11722	10584	09406	08188	06933	05640	04310
20°	0.3 28634	27483	26288	25052	23775	22458	21102	19707
21	44350	43145	41895	40602	39266	37888	36469	35010
22	59963	58704	57399	56049	54655	53217	51736	50214
23	75467	74156	72797	71391	69938	68441	66900	65315
24	90858	89495	88082	86621	85112	83556	81955	80309
25°	0.4 06131	04717	03252	01736	00171	$\bar{9}$8558	$\bar{9}$6897	$\bar{9}$5191
26	21283	19818	18301	16732	15111	13441	11723	09957
27	36308	34794	33225	31603	29928	28203	26427	24603
28	51201	49639	48020	46346	44618	42837	41006	39124
29	65960	64349	62680	60955	59175	57341	55454	53517
β = 30°	0.4 80578	78920	77203	75428	73597	71710	69770	67777
	α = 23°	24°	25°	26°	27°	28°	29°	30°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 15°$	16°	17°	18°	19°	20°	21°	22°
$\beta = 30°$	0.4 91565	90424	89214	87937	86593	85184	83712	82176
31	0.5 06354	05179	03934	02620	01238	$\bar{9}9789$	$\bar{9}8274$	$\bar{9}6695$
32	20988	19781	18501	17151	15731	14243	12687	11065
33	35465	34225	32912	31526	30069	28541	26945	25282
34	49779	48507	47161	45740	44246	42681	41045	39341
35°	0.5 63926	62623	61244	59789	58259	56657	54982	53238
36	77901	76568	75157	73668	72104	70465	68753	66969
37	91702	90339	88896	87374	85775	84101	82352	80531
38	0.6 05323	03931	02457	00903	$\bar{9}9271$	$\bar{9}7561$	$\bar{9}5777$	$\bar{9}3918$
39	18760	17339	15835	14250	12585	10842	09023	07128
40°	0.6 32010	30561	29028	27412	25715	23939	22086	20157
41	45068	43592	42030	40384	38657	36849	34963	33000
42	57931	56428	54838	53164	51406	49568	47650	45655
43	70594	69065	67449	65746	63960	62091	60143	58117
44	83054	81500	79857	78128	76314	74417	72439	70383
45°	0.6 95307	93729	92061	90306	88464	86540	84534	82450
46	0.7 07350	05748	04056	02276	00409	$\bar{9}8458$	$\bar{9}6426$	$\bar{9}4314$
47	19179	17554	15839	14034	12143	10167	08110	05973
48	30790	29144	27406	25579	23664	21665	19584	17423
49	42180	40514	38755	36905	34969	32947	30844	28660
50°	0.7 53346	51660	49881	48011	46054	44012	41887	39683
51	64285	62580	60782	58893	56917	54855	52711	50489
52	74993	73270	71454	69548	67554	65474	63313	61074
53	85467	83728	81896	79973	77962	75867	73691	71436
54	95705	93951	92103	90165	88140	86031	83841	81573
55°	0.8 05703	03935	02074	00123	$\bar{9}8084$	$\bar{9}5963$	$\bar{9}3761$	$\bar{9}1483$
56	15459	13679	11805	09842	07793	05661	03450	01163
57	24970	23179	21294	19321	17263	15123	12905	10612
58	34234	32433	30540	28559	26493	24347	22124	19827
59	43248	41439	39538	37551	35481	33331	31105	28808
$\beta = 60°$	0.8 52010	50194	48289	46297	44224	42073	39848	37553
	$\alpha = 15°$	16°	17°	18°	19°	20°	21°	22°

The function $\Lambda_0(\alpha, \beta)$

	α = 23°	24°	25°	26°	27°	28°	29°	30°
β = 30°	0.4 80578	78920	77203	75428	73597	71710	69770	67777
31	95053	93349	91584	89760	87878	85940	83947	81901
32	0.5 09379	07629	05818	03946	02015	00027	$\bar{9}$7983	$\bar{9}$5884
33	23553	21758	19901	17983	16004	13966	11872	09723
34	37569	35732	33830	31866	29840	27755	25613	23414
35°	0.5 51425	49546	47600	45591	43520	41389	39199	36953
36	65116	63195	61208	59156	57041	54865	52629	50336
37	78639	76678	74649	72555	70397	68178	65898	63561
38	91988	89988	87920	85785	83586	81325	79003	76622
39	0.6 05162	03124	01017	$\bar{9}$8843	$\bar{9}$6604	$\bar{9}$4302	$\bar{9}$1940	$\bar{8}$9518
40°	0.6 18154	16080	13936	11725	09448	07107	04705	02244
41	30963	28854	26675	24427	22113	19736	17296	14798
42	43585	41442	39228	36946	34597	32185	29710	27176
43	56015	53840	51594	49279	46897	44451	41943	39375
44	68251	66045	63768	61422	59008	56531	53991	51392
45°	0.6 80289	78054	75748	73372	70929	68422	65854	63225
46	92126	89864	87530	85126	82656	80122	77526	74871
47	0.7 03759	01471	$\bar{9}$9111	$\bar{9}$6682	$\bar{9}$4187	$\bar{9}$1627	$\bar{8}$9007	$\bar{8}$6327
48	15185	12873	10489	08036	05518	02935	00292	$\bar{9}$7591
49	26400	24066	21661	19187	16647	14044	11381	08661
50°	0.7 37403	35049	32623	30130	27572	24951	22270	19533
51	48190	45817	43375	40864	38290	35653	32958	30207
52	58758	56370	53912	51387	48799	46150	43443	40681
53	69106	66704	64233	61697	59097	56438	53722	50953
54	79231	76818	74337	71791	69183	66517	63795	61020
55°	0.7 89131	86709	84220	81667	79054	76384	73659	70883
56	98804	96375	93881	91325	88710	86038	83314	80540
57	0.8 08248	05816	03320	00763	$\bar{9}$8148	$\bar{9}$5479	$\bar{9}$2758	$\bar{8}$9990
58	17461	15028	12533	09979	07368	04705	01992	$\bar{9}$9232
59	26442	24012	21521	18972	16369	13715	11013	08267
β = 60°	0.8 35191	32766	30282	27743	25151	22510	19823	17093
	α = 23°	24°	25°	26°	27°	28°	29°	30°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 15°$	16°	17°	18°	19°	20°	21°	22°
$\beta = 60°$	0.8 52010	50194	48289	46297	44224	42073	39848	37553
61	60517	58697	56789	54796	52722	50573	48350	46060
62	68769	66946	65037	63044	60973	58828	56612	54329
63	76762	74939	73031	71042	68976	66837	64631	62360
64	84496	82675	80771	78787	76729	74601	72407	70152
65°	0.8 91969	90152	88254	86280	84233	82119	79941	77704
66	99179	97370	95481	93519	91487	89389	87232	85017
67	0.9 06126	04327	02451	00504	$\bar{9}8490$	$\bar{9}6414$	$\bar{9}4280$	$\bar{9}2093$
68	12810	11023	09164	07235	05244	03193	01087	$\bar{9}8931$
69	19229	17459	15619	13714	11748	09727	07654	05535
70°	0.9 25384	23634	21818	19940	18005	16018	13983	11904
71	31275	29549	27761	25915	24015	22068	20076	18043
72	36903	35206	33450	31641	29782	27879	25935	23954
73	42269	40606	38888	37121	35308	33455	31565	29642
74	47376	45752	44077	42357	40597	38800	36970	35110
75°	0.9 52226	50646	49021	47355	45653	43918	42154	40364
76	56823	55293	53724	52119	50481	48815	47124	45410
77	61171	59699	58192	56654	55088	53498	51886	50256
78	65276	63869	62432	60969	59481	57974	56449	54908
79	69145	67811	66452	65070	63670	62252	60821	59377
80°	0.9 72787	71534	70260	68969	67662	66343	65012	63671
81	76212	75048	73869	72676	71471	70256	69033	67803
82	79432	78367	77290	76203	75107	74005	72897	71783
83	82461	81503	80538	79565	78586	77603	76616	75626
84	85316	84475	83628	82778	81923	81066	80207	79346
85°	0.9 88015	87299	86580	85858	85135	84410	83685	82958
86	90580	89997	89412	88826	88240	87653	87066	86479
87	93035	92591	92147	91702	91258	90813	90369	89925
88	95405	95106	94807	94508	94209	93911	93613	93315
89	97717	97567	97416	97266	97116	96966	96817	96667
$\beta = 90°$	1.0 00000	00000	00000	00000	00000	00000	00000	00000
	$\alpha = 15°$	16°	17°	18°	19°	20°	21°	22°

The function $\Lambda_0(\alpha, \beta)$

		$\alpha = 23°$	24°	25°	26°	27°	28°	29°	30°
$\beta = 60°$	0.8	35191	32766	30282	27743	25151	22510	19823	17093
61		43705	41289	38816	36289	33712	31088	28420	25711
62		51985	49581	47122	44612	42054	39451	36805	34122
63		60029	57641	55201	52711	50176	47598	44980	42326
64		67838	65470	63052	60587	58079	55530	52944	50324
65°	0.8	75412	73068	70676	68240	65763	63249	60699	58117
66		82751	80435	78074	75672	73231	70755	68246	65708
67		89856	87573	85248	82884	80484	78051	75589	73099
68		96729	94483	92199	89878	87524	85139	82728	80291
69	0.9	03371	01168	$\bar{9}8929$	$\bar{9}6656$	$\bar{9}4353$	$\bar{9}2022$	$\bar{8}9667$	$\bar{8}7289$
70°	0.9	09785	07630	05441	03221	00975	$\bar{9}8703$	$\bar{9}6409$	$\bar{9}4095$
71		15974	13871	11738	09578	07393	05186	02958	00714
72		21940	19896	17825	15729	13611	11474	09319	07149
73		27689	25709	23705	21679	19635	17573	15496	13407
74		33224	31314	29384	27434	25469	23488	21495	19491
75°	0.9	38551	36718	34867	33000	31119	29226	27322	25409
76		43677	41927	40161	38383	36592	34792	32983	31167
77		48609	46948	45274	43590	41896	40194	38486	36772
78		53354	51789	50213	48629	47038	45440	43838	42232
79		57922	56459	54987	53510	52027	50539	49049	47555
80°	0.9	62323	60968	59607	58241	56872	55500	54126	52751
81		66567	65326	64082	62834	61584	60333	59080	57828
82		70666	69546	68424	67299	66174	65048	63922	62796
83		74634	73640	72645	71649	70653	69657	68662	67667
84		78485	77622	76759	75896	75034	74172	73311	72451
85°	0.9	82232	81506	80779	80054	79329	78604	77881	77159
86		85892	85306	84720	84135	83551	82968	82385	81804
87		89482	89039	88597	88156	87715	87275	86836	86398
88		93018	92721	92425	92130	91835	91540	91247	90954
89		96518	96369	96221	96073	95925	95777	95630	95483
$\beta = 90°$	1.0	00000	00000	00000	00000	00000	00000	00000	00000
		$\alpha = 23°$	24°	25°	26°	27°	28°	29°	30°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 30°$	31°	32°	33°	34°	35°	36°	37°
$\beta = 0°$	0.0 00000	00000	00000	00000	00000	00000	00000	00000
1	16304	16230	16154	16076	15995	15914	15830	15745
2	32604	32455	32303	32146	31986	31823	31655	31485
3	48894	48671	48442	48208	47968	47722	47471	47216
4	65169	64872	64567	64255	63935	63608	63274	62933
5°	0.0 81425	81054	80674	80284	79884	79476	79058	78633
6	97657	97212	96756	96289	95810	95320	94820	94310
7	0.1 13860	13342	12810	12265	11708	11137	10554	09960
8	30029	29438	28831	28209	27573	26922	26257	25578
9	46160	45496	44814	44116	43401	42670	41923	41161
10°	0.1 62247	61511	60755	59980	59187	58377	57548	56703
11	78287	77478	76648	75798	74928	74038	73129	72201
12	94273	93393	92490	91564	90617	89648	88659	87650
13	0.2 10203	09251	08275	07275	06251	05204	04135	03045
14	26070	25048	23999	22925	21825	20701	19553	18382
15°	0.2 41870	40778	39657	38510	37335	36134	34908	33657
16	57599	56437	55246	54025	52776	51499	50196	48866
17	73251	72021	70759	69466	68144	66792	65412	64004
18	88823	87525	86193	84829	83434	82008	80552	79067
19	0.3 04310	02944	01544	00109	$\bar{9}8642$	$\bar{9}7142$	$\bar{9}5612$	$\bar{9}4051$
20°	0.3 19707	18275	16806	15302	13764	12192	10587	08952
21	35010	33512	31976	30404	28795	27152	25475	23765
22	50214	48651	47049	45409	43731	42018	40269	38487
23	65315	63689	62021	60314	58569	56786	54967	53114
24	80309	78619	76888	75115	73303	71452	69565	67641
25°	0.3 95191	93440	91645	89808	87930	86013	84057	82065
26	0.4 09957	08145	06288	04388	02446	00463	$\bar{9}8441$	$\bar{9}6381$
27	24603	22731	20813	18851	16846	14799	12712	10587
28	39124	37194	35217	33194	31127	29018	26867	24677
29	53517	51530	49494	47412	45285	43115	40902	38649
$\beta = 30°$	0.4 67777	65734	63642	61502	59316	57086	54813	52500
	$\alpha = 30°$	31°	32°	33°	34°	35°	36°	37°

The function $\Lambda_0(\alpha, \beta)$

	α = 38°	39°	40°	41°	42°	43°	44°	45°
β = 0°	0.0 00000	00000	00000	00000	00000	00000	00000	00000
1	15657	15569	15479	15387	15294	15199	15104	15006
2	31311	31133	30953	30770	30583	30394	30203	30009
3	46955	46689	46419	46144	45865	45581	45294	45004
4	62585	62231	61871	61505	61133	60756	60373	59986
5°	0.0 78198	77756	77307	76849	76385	75914	75436	74953
6	93789	93259	92720	92172	91616	91051	90479	89899
7	0.1 09353	08736	08108	07470	06822	06164	05497	04822
8	24887	24182	23466	22737	21998	21248	20487	19717
9	40384	39593	38789	37971	37141	36299	35445	34581
10°	0.1 55842	54965	54073	53167	52246	51313	50367	49408
11	71256	70294	69315	68320	67310	66286	65248	64197
12	86621	85574	84509	83427	82328	81214	80085	78942
13	0.2 01934	00803	$\bar{9}$9652	$\bar{9}$8483	$\bar{9}$7297	$\bar{9}$6093	$\bar{9}$4874	$\bar{9}$3639
14	17189	15975	14740	13485	12211	10919	09611	08286
15°	0.2 32383	31086	29767	28428	27068	25689	24292	22878
16	47512	46133	44732	43308	41863	40397	38913	37411
17	62570	61111	59628	58121	56592	55042	53471	51882
18	77555	76016	74452	72863	71251	69617	67962	66287
19	92462	90845	89201	87531	85837	84121	82382	80622
20°	0.3 07286	05592	03869	02121	00346	$\bar{9}$8548	$\bar{9}$6727	$\bar{9}$4884
21	22025	20254	18454	16627	14774	12896	10994	09070
22	36673	34827	32952	31048	29117	27160	25179	23175
23	51227	49308	47358	45379	43371	41338	39279	37197
24	65683	63691	61669	59616	57534	55425	53290	51132
25°	0.3 80037	77975	75880	73755	71600	69418	67209	64976
26	94285	92154	89990	87794	85568	83314	81033	78726
27	0.4 08424	06225	03993	01728	$\bar{9}$9433	$\bar{9}$7109	$\bar{9}$4757	$\bar{9}$2380
28	22449	20185	17886	15555	13192	10799	08379	05934
29	36358	34030	31667	29270	26841	24383	21897	19384
β = 30°	0.4 50147	47756	45330	42870	40378	37856	35306	32729
	α = 38°	39°	40°	41°	42°	43°	44°	45°

The function $A_\theta(\alpha, \beta)$

	$\alpha = 30°$	31°	32°	33°	34°	35°	36°	37°
$\beta = 30°$	0.4 67777	65734	63642	61502	59316	57086	54813	52500
31	81901	79803	77655	75459	73216	70928	68597	66224
32	95884	93733	91532	89281	86982	84638	82250	79820
33	0.5 09723	07521	05266	02962	00610	$\bar{9}8211$	$\bar{9}5768$	$\bar{9}3282$
34	23414	21161	18856	16500	14096	11645	09149	06609
35°	0.5 36953	34652	32297	29892	27437	24935	22388	19797
36	50336	47988	45586	43133	40630	38079	35483	32843
37	63561	61167	58720	56220	53671	51073	48430	45743
38	76622	74185	71694	69150	66556	63915	61227	58495
39	89518	87039	84506	81920	79284	76600	73870	71096
40°	0.6 02244	$\bar{9}9726$	$\bar{9}7153$	$\bar{9}4527$	$\bar{9}1851$	$\bar{8}9127$	$\bar{8}6356$	$\bar{8}3543$
41	14798	12242	09631	06968	04254	01492	$\bar{9}8684$	$\bar{9}5833$
42	27176	24584	21938	19239	16490	13692	10850	07964
43	39375	36750	34070	31338	28556	25726	22851	19933
44	51392	48736	46026	43263	40450	37590	34686	31738
45°	0.6 63225	60541	57801	55010	52170	49283	46351	43378
46	74871	72160	69395	66578	63713	60801	57846	54849
47	86327	83592	80803	77964	75076	72143	69167	66150
48	97591	94835	92025	89166	86259	83307	80313	77279
49	0.7 08661	05885	03058	00182	$\bar{9}7258$	$\bar{9}4291$	$\bar{9}1282$	$\bar{8}8235$
50°	0.7 19533	16742	13900	11009	08073	05094	02074	$\bar{9}9016$
51	30207	27404	24549	21648	18701	15713	12685	09621
52	40681	37867	35004	32095	29142	26148	23116	20049
53	50953	48132	45263	42350	39394	36398	33365	30298
54	61020	58196	55325	52411	49455	46461	43431	40369
55°	0.7 70883	68059	65190	62278	59326	56337	53314	50260
56	80540	77719	74855	71949	69005	66026	63014	59972
57	89990	87176	84320	81425	78493	75527	72530	69504
58	99232	96429	93586	90704	87788	84840	81862	78856
59	0.8 08267	05478	02651	$\bar{9}9788$	$\bar{9}6892$	$\bar{9}3965$	$\bar{9}1010$	$\bar{8}8030$
$\beta = 60°$	0.8 17093	14323	11517	08676	05804	02903	$\bar{9}9976$	$\bar{9}7024$
	$\alpha = 30°$	31°	32°	33°	34°	35°	36°	37°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 38°$	39°	40°	41°	42°	43°	44°	45°
$\beta = 30°$	0.4 50147	47756	45330	42870	40378	37856	35306	32729
31	63811	61361	58874	56353	53799	51216	48603	45964
32	77349	74840	72295	69714	67102	64458	61786	59088
33	90756	88191	85589	82952	80282	77582	74852	72097
34	0.5 04029	01410	$\bar{9}8754$	$\bar{9}6062$	$\bar{9}3338$	$\bar{9}0583$	$\bar{8}7799$	$\bar{8}4988$
35°	0.5 17165	14494	11786	09042	06266	03459	00622	$\bar{9}7760$
36	30161	27441	24683	21890	19063	16207	13321	10409
37	43015	40247	37441	34601	31728	28824	25892	22933
38	55722	52909	50059	47174	44257	41309	38333	35330
39	68280	65425	62534	59607	56648	53658	50641	47598
40°	0.5 80687	77793	74862	71896	68898	65870	62815	59735
41	92940	90009	87041	84040	81006	77943	74853	71738
42	0.6 05037	02072	$\bar{9}9070$	$\bar{9}6035$	$\bar{9}2969$	$\bar{8}9874$	$\bar{8}6752$	$\bar{8}3606$
43	16975	13978	10946	07881	04785	01661	$\bar{9}8510$	$\bar{9}5337$
44	28751	25727	22667	19575	16453	13303	10127	06929
45°	0.6 40365	37315	34231	31115	27970	24797	21600	18381
46	51814	48742	45637	42501	39335	36144	32929	29692
47	63095	60005	56882	53729	50547	47341	44111	40860
48	74208	71103	67966	64799	61605	58386	55146	51885
49	85151	82034	78886	75709	72506	69280	66032	62766
50°	0.6 95923	92798	89642	86459	83251	80020	76769	73501
51	0.7 06522	03392	00233	$\bar{9}7048$	$\bar{9}3838$	$\bar{9}0607$	$\bar{8}7357$	$\bar{8}4090$
52	16948	13817	10658	07474	04267	01040	$\bar{9}7794$	$\bar{9}4533$
53	27199	24071	20917	17738	14538	11318	08081	04830
54	37276	34155	31008	27839	24649	21441	18218	14980
55°	0.7 47177	44066	40932	37777	34602	31410	28203	24985
56	56902	53807	50689	47551	44395	41224	38039	34843
57	66452	63376	60279	57163	54031	50884	47725	44556
58	75826	72774	69702	66613	63508	60390	57261	54124
59	85026	82002	78959	75901	72828	69744	66650	63548
$\beta = 60°$	0.7 94052	91060	88051	85028	81992	78946	75891	72830
	$\alpha = 38°$	39°	40°	41°	42°	43°	44°	45°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 30°$	31°	32°	33°	34°	35°	36°	37°
$\beta = 60°$	0.8 17093	14323	11517	08676	05804	02903	$\bar{9}9976$	$\bar{9}7024$
61	25711	22964	20183	17369	14525	11654	08759	05842
62	34122	31402	28650	25867	23056	20220	17362	14483
63	42326	39638	36919	34171	31398	28602	25785	22949
64	50324	47672	44991	42284	39554	36801	34030	31242
65°	0.8 58117	55506	52869	50207	47523	44820	42100	39364
66	65708	63143	60553	57941	55310	52660	49996	47317
67	73099	70584	68047	65490	62916	60325	57721	55105
68	80291	77832	75353	72857	70344	67817	65278	62730
69	87289	84891	82475	80043	77598	75140	72672	70195
70°	0.8 94095	91763	89416	87054	84681	82297	79904	77505
71	0.9 00714	$\bar{9}8453$	$\bar{9}6179$	$\bar{9}3893$	$\bar{9}1597$	$\bar{8}9292$	$\bar{8}6980$	$\bar{8}4663$
72	07149	04966	02771	00565	$\bar{9}8352$	$\bar{9}6131$	$\bar{9}3905$	$\bar{9}1674$
73	13407	11306	09195	07075	04949	02818	00682	$\bar{9}8543$
74	19491	17478	15457	13429	11395	09358	07317	05275
75°	0.9 25409	23489	21563	19631	17695	15757	13817	11876
76	31167	29346	27519	25689	23856	22022	20187	18351
77	36772	35055	33333	31610	29884	28159	26433	24708
78	42232	40623	39012	37400	35787	34174	32562	30952
79	47555	46060	44564	43068	41572	40077	38583	37091
80°	0.9 52751	51374	49998	48622	47247	45873	44502	43132
81	57828	56575	55322	54071	52821	51573	50327	49083
82	62796	61671	60546	59424	58302	57183	56066	54952
83	67667	66673	65680	64690	63701	62714	61729	60748
84	72451	71592	70735	69879	69025	68174	67325	66478
85°	0.9 77159	76439	75719	75002	74286	73573	72861	72152
86	81804	81225	80646	80069	79494	78920	78349	77779
87	86398	85962	85526	85092	84658	84227	83796	83368
88	90954	90662	90370	90080	89790	89502	89214	88928
89	95483	95337	95191	95046	94901	94756	94612	94469
$\beta = 90°$	1.0 00000	00000	00000	00000	00000	00000	00000	00000
	$\alpha = 30°$	31°	32°	33°	34°	35°	36°	37°

The function $\Delta_0(\alpha, \beta)$

	$\alpha = 38°$	39°	40°	41°	42°	43°	44°	45°
$\beta = 60°$	0.7 94052	91060	88051	85028	81992	78946	75891	72830
61	0.8 02905	$\bar{9}9950$	$\bar{9}6980$	$\bar{9}3996$	$\bar{9}1001$	$\bar{8}7998$	$\bar{8}4987$	$\bar{8}1970$
62	11586	08672	05745	02806	$\bar{9}9857$	$\bar{9}6901$	$\bar{9}3938$	$\bar{9}0971$
63	20096	17229	14350	11460	08562	05657	02747	$\bar{9}9834$
64	28439	25623	22796	19960	17116	14268	11416	08561
65°	0.8 36615	33855	31085	28307	25524	22736	19946	17155
66	44627	41927	39220	36505	33785	31065	28341	25618
67	52479	49844	47203	44557	41907	39256	36603	33952
68	60172	57608	55038	52465	49889	47312	44736	42162
69	67711	65222	62728	60232	57735	55238	52742	50249
70°	0.8 75099	72690	70277	67863	65449	63036	60625	58217
71	82341	80016	77689	75362	73035	70711	68389	66071
72	89440	87205	84968	82732	80498	78266	76037	73814
73	96402	94260	92119	89979	87841	85706	83575	81450
74	0.9 03232	01189	$\bar{9}9147$	$\bar{9}7107$	$\bar{9}5069$	$\bar{9}3036$	$\bar{9}1007$	$\bar{8}8983$
75°	0.9 09935	07995	06056	04121	02188	00260	$\bar{9}8337$	$\bar{9}6419$
76	16517	14684	12854	11027	09203	07384	05570	03762
77	22984	21263	19545	17830	16119	14413	12712	11018
78	29344	27738	26135	24537	22942	21353	19769	18190
79	35602	34115	32632	31153	29679	28209	26745	25286
80°	0.9 41766	40402	39042	37686	36335	34988	33647	32311
81	47843	46606	45372	44142	42917	41696	40481	39270
82	53841	52733	51629	50528	49432	48340	47253	46170
83	59769	58793	57821	56852	55887	54926	53969	53017
84	65634	64793	63955	63120	62289	61461	60637	59817
85°	0.9 71445	70741	70039	69340	68644	67952	67262	66576
86	77211	76645	76082	75521	74962	74406	73852	73302
87	82940	82515	82091	81669	81249	80830	80414	80000
88	88642	88358	88075	87793	87512	87233	86955	86678
89	94326	94183	94041	93900	93760	93620	93481	93343
$\beta = 90°$	1.0 00000	00000	00000	00000	00000	00000	00000	00000
	$\alpha = 38°$	39°	40°	41°	42°	43°	44°	45°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 45°$	46°	47°	48°	49°	50°	51°	52°
$\beta =$ 0°	0.0 00000	00000	00000	00000	00000	00000	00000	00000
1	15006	14908	14809	14709	14607	14505	14403	14299
2	30009	29813	29614	29414	29211	29007	28802	28594
3	45004	44709	44412	44111	43808	43502	43194	42883
4	59986	59594	59198	58797	58393	57986	57575	57162
5°	0.0 74953	74463	73968	73469	72964	72455	71943	71426
6	89899	89313	88720	88121	87517	86907	86293	85674
7	0.1 04822	04139	03449	02751	02047	01337	00622	$\bar{9}9902$
8	19717	18938	18151	17355	16552	15742	14927	14105
9	34581	33706	32822	31929	31028	30119	29204	28282
10°	0.1 49408	48439	47459	46470	45471	44464	43449	42428
11	64197	63133	62058	60973	59878	58773	57661	56541
12	78942	77785	76616	75435	74244	73043	71834	70616
13	93639	92390	91128	89854	88568	87271	85966	84652
14	0.2 08286	06946	05591	04224	02844	01454	00053	$\bar{9}8643$
15°	0.2 22878	21447	20002	18543	17070	15587	14092	12589
16	37411	35891	34356	32806	31243	29667	28080	26484
17	51882	50274	48651	47011	45358	43692	42014	40326
18	66287	64593	62882	61155	59413	57658	55890	54113
19	80622	78843	77046	75233	73404	71561	69706	67840
20°	0.2 94884	93022	91141	89242	87328	85399	83458	81505
21	0.3 09070	07125	05161	03179	01181	$\bar{9}9169$	$\bar{9}7143$	$\bar{9}5105$
22	23175	21150	19105	17042	14962	12866	10758	08638
23	37197	35093	32968	30825	28665	26490	24300	22099
24	51132	48950	46749	44528	42289	40035	37767	35487
25°	0.3 64976	62720	60442	58145	55830	53500	51155	48799
26	78726	76397	74046	71675	69286	66882	64463	62031
27	92380	89979	87557	85114	82654	80177	77686	75183
28	0.4 05934	03464	00972	$\bar{9}8460$	$\bar{9}5930$	$\bar{9}3383$	$\bar{9}0822$	$\bar{8}8250$
29	19384	16848	14289	11709	09112	06498	03870	01230
$\beta =$ 30°	0.4 32729	30127	27504	24860	22197	19519	16826	14121
	$\alpha = 45°$	46°	47°	48°	49°	50°	51°	52°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 53°$	54°	55°	56°	57°	58°	59°	60°
$\beta = 0°$	0.0 00000	00000	00000	00000	00000	00000	00000	00000
1	14195	14090	13985	13879	13774	13668	13562	13456
2	28386	28177	27966	27756	27544	27332	27121	26909
3	42571	42257	41942	41626	41309	40992	40674	40357
4	56746	56328	55908	55487	55065	54642	54219	53797
5°	0.0 70907	70385	69861	69336	68809	68281	67754	67226
6	85052	84427	83799	83170	82539	81907	81275	80643
7	99177	98450	97719	96986	96251	95515	94779	94044
8	0.1 13280	12450	11616	10780	09942	09104	08265	07426
9	27355	26424	25489	24551	23611	22670	21728	20788
10°	0.1 41401	40370	39334	38295	37253	36211	35168	34126
11	55415	54283	53147	52008	50867	49724	48581	47439
12	69392	68162	66927	65689	64448	63206	61964	60723
13	83330	82003	80671	79334	77996	76655	75315	73976
14	97226	95803	94374	92941	91505	90068	88632	87196
15°	0.2 11077	09558	08034	06506	04975	03443	01911	00380
16	24879	23267	21649	20027	18402	16776	15151	13527
17	38630	36926	35216	33501	31784	30066	28348	26632
18	52326	50531	48731	46926	45118	43309	41501	39695
19	65965	64081	62192	60298	58402	56504	54607	52713
20°	0.2 79543	77573	75597	73616	71632	69648	67664	65684
21	93058	91003	88942	86876	84807	82738	80669	78605
22	0.3 06508	04369	02225	00075	$\bar{9}7924$	$\bar{9}5772$	$\bar{9}3621$	$\bar{9}1474$
23	19888	17669	15443	13213	10980	08748	06517	04290
24	33197	30899	28594	26285	23974	21663	19354	17049
25°	0.3 46432	44057	41676	39290	36903	34516	32131	29751
26	59590	57141	54685	52226	49765	47304	44846	42394
27	72669	70148	67621	65089	62558	60025	57497	54974
28	85667	83076	80479	77879	75278	72678	70081	67491
29	98580	95922	93259	90592	87925	85260	82598	79943
$\beta = 30°$	0.4 11407	08685	05958	03228	00497	$\bar{9}7769$	$\bar{9}5045$	$\bar{9}2328$
	$\alpha = 53°$	54°	55°	56°	57°	58°	59°	60°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 45°$	46°	47°	48°	49°	50°	51°	52°
$\beta = 30°$	0.4 32729	30127	27504	24860	22197	19519	16826	14121
31	45964	43301	40615	37908	35184	32443	29688	26921
32	59088	56364	53619	50853	48068	45268	42453	39628
33	72097	69316	66513	63690	60848	57991	55120	52238
34	84988	82153	79295	76418	73522	70611	67687	64751
35°	0.4 97760	94873	91963	89034	86087	83125	80150	77164
36	0.5 10409	07473	04514	01537	$\bar{9}8541$	$\bar{9}5531$	$\bar{9}2508$	$\bar{8}9475$
37	22933	19951	16947	13923	10883	07827	04760	01683
38	35330	32305	29257	26191	23109	20012	16903	13785
39	47598	44532	41445	38340	35218	32083	28936	25781
40°	0.5 59735	56632	53508	50366	47209	44038	40857	37668
41	71738	68601	65444	62269	59080	55877	52665	49445
42	83606	80438	77251	74047	70828	67598	64358	61111
43	95337	92142	88928	85698	82454	79199	75936	72665
44	0.6 06929	03710	00473	$\bar{9}7221$	$\bar{9}3956$	$\bar{9}0680$	$\bar{8}7396$	$\bar{8}4106$
45°	0.6 18381	15142	11886	08615	05332	02038	$\bar{9}8738$	$\bar{9}5432$
46	29692	26437	23165	19879	16581	13274	09961	06644
47	40860	37592	34308	31011	27703	24387	21065	17739
48	51885	48608	45316	42011	38697	35375	32048	28718
49	62766	59483	56186	52878	49561	46238	42910	39581
50°	0.6 73501	70217	66920	63613	60297	56976	53651	50326
51	84090	80809	77516	74213	70903	67589	64271	60954
52	94533	91259	87974	84680	81380	78076	74770	71465
53	0.7 04830	01567	$\bar{9}8293$	$\bar{9}5013$	$\bar{9}1726$	$\bar{8}8437$	$\bar{8}5148$	$\bar{8}1859$
54	14980	11732	08475	05212	01944	$\bar{9}8674$	$\bar{9}5404$	$\bar{9}2137$
55°	0.7 24985	21756	18519	15277	12032	08785	05540	02298
56	34843	31638	28426	25210	21991	18773	15556	12343
57	44556	41379	38196	35010	31823	28636	25452	22273
58	54124	50979	47831	44679	41528	38377	35231	32090
59	63548	60441	57330	54218	51106	47997	44892	41793
$\beta = 60°$	0.7 72830	69764	66696	63627	60560	57496	54437	51385
	$\alpha = 45°$	46°	47°	48°	49°	50°	51°	52°

The function $\Lambda_0(\alpha, \beta)$

	α = 53°	54°	55°	56°	57°	58°	59°	60°
β = 30°	0.4 11407	08685	05958	03228	00497	$\bar{9}$7769	$\bar{9}$5045	$\bar{9}$2328
31	24145	21362	18573	15783	12992	10203	07420	04644
32	36793	33951	31104	28255	25407	22562	19722	16890
33	49347	46450	43548	40644	37741	34842	31949	29064
34	61807	58857	55903	52947	49993	47043	44100	41166
35°	0.4 74170	71170	68167	65163	62161	59163	56173	53192
36	86434	83388	80339	77290	74243	71201	68167	65143
37	98598	95509	92418	89326	86238	83155	80081	77017
38	0.5 10661	07532	04401	01271	$\bar{9}$8144	$\bar{9}$5024	$\bar{9}$1913	$\bar{8}$8813
39	22619	19454	16287	13122	09961	06807	03662	00530
40°	0.5 34473	31275	28076	24879	21687	18502	15328	12167
41	46220	42993	39765	36540	33321	30110	26909	23722
42	57860	54607	51354	48105	44862	41628	38405	35196
43	69391	66116	62843	59573	56310	53056	49815	46588
44	80813	77520	74229	70942	67663	64394	61137	57897
45°	0.5 92124	88817	85512	82212	78921	75640	72373	69122
46	0.6 03325	00006	$\bar{9}$6692	$\bar{9}$3383	$\bar{9}$0083	$\bar{8}$6795	$\bar{8}$3521	$\bar{8}$0263
47	14413	11088	07768	04454	01150	$\bar{9}$7858	$\bar{9}$4580	$\bar{9}$1320
48	25389	22062	18739	15425	12120	08828	05552	02293
49	36252	32926	29606	26295	22994	19706	16434	13181
50°	0.6 47002	43682	40369	37064	33771	30491	27229	23985
51	57639	54329	51026	47733	44451	41184	37935	34705
52	68164	64867	61579	58301	55035	51785	48552	45340
53	78575	75297	72027	68769	65523	62294	59082	55891
54	88874	85618	82372	79137	75916	72711	69525	66359
55°	0.6 99061	95832	92612	89405	86213	83037	79880	76745
56	0.7 09136	05938	02750	$\bar{9}$9575	$\bar{9}$6415	$\bar{9}$3272	$\bar{9}$0149	$\bar{8}$7048
57	19101	15938	12786	09647	06524	03419	00333	$\bar{9}$7269
58	28956	25833	22721	19622	16540	13476	10432	07410
59	38703	35623	32555	29502	26464	23446	20447	17472
β = 60°	0.7 48342	45310	42291	39286	36298	33329	30380	27455
	α = 53°	54°	55°	56°	57°	58°	59°	60°

The function $\Lambda_0(\alpha, \beta)$

	α =	45°	46°	47°	48°	49°	50°	51°	52°
β = 60°	0.7	72830	69764	66696	63627	60560	57496	54437	51385
61		81970	78951	75929	72909	69890	66876	63867	60866
62		90971	88002	85032	82064	79099	76139	73185	70239
63		99834	96920	94007	91095	88188	85286	82391	79505
64	0.8	08561	05707	02854	00004	$\bar{9}7158$	$\bar{9}4319$	$\bar{9}1488$	$\bar{8}8666$
65°	0.8	17155	14365	11577	08792	06013	03241	00478	$\bar{9}7724$
66		25618	22896	20177	17463	14755	12054	09363	06681
67		33952	31304	28659	26019	23386	20761	18145	15540
68		42162	39590	37024	34463	31909	29364	26828	24303
69		50249	47759	45275	42797	40327	37865	35414	32973
70°	0.8	58217	55814	53417	51026	48643	46269	43906	41553
71		66071	63758	61451	59152	56861	54579	52307	50047
72		73814	71595	69384	67179	64983	62797	60621	58456
73		81450	79330	77217	75111	73015	70927	68851	66785
74		88983	86966	84955	82953	80959	78974	77000	75037
75°	0.8	96419	94508	92604	90708	88820	86942	85073	83216
76	0.9	03762	01961	00166	$\bar{9}8380$	$\bar{9}6602$	$\bar{9}4833$	$\bar{9}3074$	$\bar{9}1326$
77		11018	09329	07648	05975	04310	02653	01007	$\bar{9}9370$
78		18190	16619	15054	13497	11947	10406	08875	07353
79		25286	23834	22389	20950	19520	18097	16684	15279
80°	0.9	32311	30981	29658	28341	27032	25731	24437	23152
81		39270	38066	36867	35675	34489	33311	32140	30977
82		46170	45093	44021	42956	41896	40843	39797	38758
83		53017	52069	51127	50190	49258	48333	47413	46500
84		59817	59001	58190	57383	56581	55784	54993	54207
85°	0.9	66576	65894	65215	64540	63870	63204	62542	61885
86		73302	72754	72210	71668	71130	70596	70065	69538
87		80000	79588	79179	78772	78368	77966	77567	77171
88		86678	86403	86130	85858	85588	85320	85053	84789
89		93343	93205	93068	92932	92797	92663	92529	92397
β = 90°	1.0	00000	00000	00000	00000	00000	00000	00000	00000
	α =	45°	46°	47°	48°	49°	50°	51°	52°

The function $\Lambda_0(\alpha, \beta)$

	α =	53°	54°	55°	56°	57°	58°	59°	60°
β = 60°	0.7	48342	45310	42291	39286	36298	33329	30380	27455
61		57875	54895	51929	48977	46043	43127	40232	37360
62		67304	64380	61471	58576	55699	52841	50004	47190
63		76630	73767	70918	68085	65269	62473	59697	56945
64		85855	83057	80273	77505	74755	72024	69314	66627
65°	0.7	94982	92252	89537	86839	84158	81496	78855	76237
66	0.8	04011	01355	$\bar{9}8713$	$\bar{9}6087$	$\bar{9}3480$	$\bar{9}0891$	$\bar{8}8323$	$\bar{8}5778$
67		12947	10367	07802	05254	02723	00211	$\bar{9}7720$	$\bar{9}5251$
68		21791	19292	16808	14340	11889	09458	07047	04658
69		30545	28131	25731	23348	20981	18634	16307	14001
70°	0.8	39214	36887	34576	32280	30002	27742	25502	23283
71		47799	45564	43344	41140	38953	36784	34634	32504
72		56304	54165	52040	49931	47838	45763	43706	41669
73		64732	62691	60665	58654	56658	54680	52720	50780
74		73086	71148	69223	67313	65418	63540	61680	59837
75°	0.8	81370	79537	77717	75911	74120	72345	70587	68846
76		89589	87863	86151	84452	82767	81098	79444	77807
77		97744	96130	94528	92939	91363	89801	88255	86724
78	0.9	05841	04341	02851	01374	$\bar{9}9910$	$\bar{9}8459$	$\bar{9}7022$	$\bar{9}5600$
79		13884	12499	11125	09763	08412	07074	05749	04438
80°	0.9	21876	20610	19353	18108	16873	15649	14438	13240
81		29822	28676	27540	26412	25295	24189	23093	22010
82		37726	36703	35688	34681	33683	32696	31718	30750
83		45593	44694	43802	42917	42041	41173	40314	39464
84		53427	52653	51886	51125	50371	49625	48886	48155
85°	0.9	61233	60586	59944	59309	58679	58055	57437	56826
86		69015	68496	67981	67472	66966	66466	65971	65481
87		76778	76388	76001	75618	75238	74862	74490	74122
88		84527	84266	84008	83752	83499	83248	83000	82754
89		92266	92135	92006	91878	91751	91626	91502	91379
β = 90°	1.0	00000	00000	00000	00000	00000	00000	00000	00000

	α =	53°	54°	55°	56°	57°	58°	59°	60°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 60°$	61°	62°	63°	64°	65°	66°	67°
$\beta = 0°$	0.0 00000	00000	00000	00000	00000	00000	00000	00000
1	13456	13350	13245	13139	13035	12931	12828	12726
2	26909	26697	26487	26277	26068	25860	25654	25450
3	40357	40040	39724	39409	39096	38785	38476	38170
4	53797	53375	52954	52535	52118	51703	51292	50884
5°	0.0 67226	66700	66175	65651	65131	64614	64100	63592
6	80643	80012	79383	78756	78133	77514	76899	76289
7	94044	93309	92577	91848	91122	90401	89685	88976
8	0.1 07426	06589	05754	04923	04096	03274	02458	01650
9	20788	19849	18912	17980	17052	16130	15215	14309
10°	0.1 34126	33086	32049	31016	29989	28968	27955	26951
11	47439	46299	45162	44030	42904	41785	40675	39575
12	60723	59484	58249	57019	55795	54580	53374	52179
13	73976	72640	71308	69981	68661	67350	66049	64761
14	87196	85764	84336	82913	81499	80094	78700	77319
15°	0.2 00380	$\bar{9}8853$	$\bar{9}7331$	$\bar{9}5815$	$\bar{9}4307$	$\bar{9}2809$	$\bar{9}1324$	$\bar{8}9852$
16	13527	11906	10291	08683	07083	05495	03919	02358
17	26632	24920	23214	21515	19826	18148	16484	14835
18	39695	37893	36098	34310	32533	30767	29016	27282
19	52713	50823	48940	47066	45202	43351	41515	39697
20°	0.2 65684	63708	61739	59780	57832	55897	53979	52079
21	78605	76545	74493	72451	70420	68404	66405	64426
22	91474	89333	87199	85076	82966	80871	78793	76736
23	0.3 04290	02069	$\bar{9}9856$	$\bar{9}7655$	$\bar{9}5467$	$\bar{9}3295$	$\bar{9}1141$	$\bar{8}9009$
24	17049	14751	12462	10185	07921	05675	03447	01242
25°	0.3 29751	27379	25015	22664	20328	18009	15710	13435
26	42394	39949	37514	35092	32685	30296	27929	25585
27	54974	52459	49956	47465	44990	42535	40101	37693
28	67491	64910	62339	59783	57243	54724	52227	49756
29	79943	77297	74663	72044	69442	66861	64304	61773
$\beta = 30°$	0.3 92328	89621	86926	84247	81586	78946	76331	73743
	$\alpha = 60°$	61°	62°	63°	64°	65°	66°	67°

The function $\Lambda_0(\alpha, \beta)$

	α = 68°	69°	70°	71°	72°	73°	74°	75°
β = 0°	0.0 00000	00000	00000	00000	00000	00000	00000	00000
1	12625	12525	12426	12329	12234	12140	12049	11960
2	25248	25048	24851	24657	24466	24279	24097	23919
3	37867	37568	37272	36982	36696	36416	36142	35876
4	50481	50082	49689	49302	48922	48549	48185	47829
5°	0.0 63088	62591	62100	61617	61143	60677	60223	59779
6	75686	75091	74503	73925	73357	72800	72255	71724
7	88274	87581	86897	86224	85563	84914	84281	83663
8	0.1 00850	00060	$\bar{9}9280$	$\bar{9}8513$	$\bar{9}7760$	$\bar{9}7021$	$\bar{9}6299$	$\bar{9}5594$
9	13412	12525	11651	10791	09946	09118	08308	07518
10°	0.1 25958	24976	24009	23056	22121	21204	20307	19433
11	38486	37411	36351	35307	34282	33278	32295	31338
12	50996	49828	48677	47543	46430	45339	44272	43232
13	63486	62226	60984	59762	58562	57385	56235	55114
14	75953	74603	73273	71963	70677	69416	68184	66983
15°	0.1 88396	86958	85540	84145	82774	81431	80119	78839
16	0.2 00814	$\bar{9}9289$	$\bar{9}7785$	$\bar{9}6305$	$\bar{9}4852$	$\bar{9}3429$	$\bar{9}2037$	$\bar{9}0680$
17	13204	11594	10007	08444	06910	05407	03938	02506
18	25567	23873	22203	20560	18947	17366	15821	14316
19	37899	36124	34374	32652	30961	29305	27686	26108
20°	0.2 50200	48345	46517	44718	42952	41221	39531	37883
21	62468	60536	58631	56757	54918	53116	51355	49638
22	74702	72694	70716	68769	66858	64986	63157	61375
23	86901	84820	82769	80752	78772	76832	74937	73091
24	99062	96910	94790	92705	90658	88653	86694	84785
25°	0.3 11185	08965	06778	04626	02515	00447	$\bar{9}8427$	$\bar{9}6459$
26	23269	20983	18731	16516	14343	12214	10135	08109
27	35312	32963	30649	28373	26140	23954	21818	19737
28	47313	44904	42530	40197	37907	35664	33474	31341
29	59272	56805	54374	51985	49641	47345	45104	42920
β = 30°	0.3 71186	68664	66180	63738	61342	58997	56706	54475
	α = 68°	69°	70°	71°	72°	73°	74°	75°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 60°$	61°	62°	63°	64°	65°	66°	67°
$\beta = 30°$	0.3 92328	89621	86926	84247	81586	78946	76331	73743
31	0.4 04644	01879	$\bar{9}9126$	$\bar{9}6390$	$\bar{9}3673$	$\bar{9}0977$	$\bar{8}8307$	$\bar{8}5666$
32	16890	14069	11262	08472	05701	02954	00232	$\bar{9}7539$
33	29064	26191	23333	20492	17671	14874	12103	09363
34	41166	38244	35337	32448	29580	26736	23920	21135
35°	0.4 53192	50225	47272	44339	41428	38541	35683	32856
36	65143	62133	59139	56165	53213	50287	47389	44525
37	77017	73968	70936	67924	64935	61972	59039	56140
38	88813	85728	82661	79615	76592	73597	70632	67701
39	0.5 00530	$\bar{9}7413$	$\bar{9}4314$	$\bar{9}1237$	$\bar{8}8185$	$\bar{8}5160$	$\bar{8}2166$	$\bar{7}9208$
40°	0.5 12167	09021	05895	02791	$\bar{9}9711$	$\bar{9}6661$	$\bar{9}3642$	$\bar{9}0659$
41	23722	20552	17402	14274	11172	08099	05059	02055
42	35196	32005	28834	25686	22565	19474	16416	13394
43	46588	43379	40192	37028	33891	30785	27712	24677
44	57897	54674	51474	48298	45149	42032	38948	35903
45°	0.5 69122	65890	62680	59495	56339	53214	50124	47072
46	80263	77025	73810	70621	67460	64332	61238	58184
47	91320	88081	84864	81674	78513	75384	72292	69238
48	0.6 02293	$\bar{9}9055$	$\bar{9}5842$	$\bar{9}2654$	$\bar{8}9497$	$\bar{8}6372$	$\bar{8}3284$	$\bar{8}0235$
49	13181	09950	06742	03562	00412	$\bar{9}7295$	$\bar{9}4215$	$\bar{9}1174$
50°	0.6 23985	20763	17567	14397	11258	08153	05085	02057
51	34705	31497	28314	25160	22036	18947	15894	12882
52	45340	42150	38986	35851	32746	29676	26643	23650
53	55891	52724	49582	46469	43388	40341	37331	34362
54	66359	63218	60103	57017	53962	50942	47959	45017
55°	0.6 76745	73633	70549	67493	64469	61480	58528	55617
56	87048	83971	80920	77899	74910	71955	69038	66162
57	97269	94230	91218	88236	85285	82369	79490	76652
58	0.7 07410	04414	01444	$\bar{9}8503$	$\bar{9}5595$	$\bar{9}2721$	$\bar{8}9885$	$\bar{8}7088$
59	17472	14521	11597	08703	05841	03013	00222	$\bar{9}7471$
$\beta = 60°$	0.7 27455	24554	21680	18836	16024	13246	10504	07802
	$\alpha = 60°$	61°	62°	63°	64°	65°	66°	67°

The function $\Lambda_0(\alpha, \beta)$

	α = 68°	69°	70°	71°	72°	73°	74°	75°
β = 30°	0.3 71186	68664	66180	63738	61342	58997	56706	54475
31	83056	80482	77947	75455	73010	70617	68280	66004
32	94879	92256	89673	87134	84644	82206	79825	77508
33	0.4 06656	03987	01359	$\bar{9}$8776	$\bar{9}$6242	$\bar{9}$3763	$\bar{9}$1342	$\bar{8}$8984
34	18385	15673	13003	10379	07805	05287	02828	00434
35°	0.4 30065	27313	24604	21943	19332	16778	14284	11857
36	41696	38908	36163	33467	30823	28235	25710	23251
37	53278	50456	47679	44951	42276	39659	37104	34618
38	64808	61956	59150	56394	53691	51048	48467	45956
39	76287	73409	70577	67796	65069	62402	59799	57266
40°	0.4 87715	84813	81959	79156	76408	73720	71098	68546
41	99090	96169	93296	90474	87708	85004	82365	79797
42	0.5 10413	07476	04587	01750	$\bar{9}$8970	$\bar{9}$6251	$\bar{9}$3599	$\bar{9}$1019
43	21682	18733	15832	12983	10192	07463	04801	02211
44	32899	29940	27030	24174	21375	18639	15970	13373
45°	0.5 44062	41097	38183	35321	32519	29778	27106	24506
46	55171	52205	49289	46426	43622	40881	38208	35609
47	66227	63262	60348	57488	54686	51948	49278	46682
48	77229	74269	71361	68506	65711	62979	60315	57725
49	88177	85227	82327	79482	76696	73973	71319	68738
50°	0.5 99072	96134	93247	90415	87641	84932	82290	79721
51	0.6 09913	06991	04121	01305	$\bar{9}$8548	$\bar{9}$5854	$\bar{9}$3228	$\bar{9}$0675
52	20701	17799	14948	12152	09415	06741	04134	01600
53	31436	28558	25730	22957	20243	17592	15008	12496
54	42119	39268	36467	33721	31033	28407	25849	23362
55°	0.6 52749	49929	47159	44443	41784	39188	36659	34200
56	63329	60542	57806	55123	52498	49935	47437	45010
57	73857	71108	68409	65763	63175	60647	58184	55791
58	84335	81627	78969	76363	73814	71325	68901	66545
59	94763	92100	89486	86924	84418	81971	79587	77272
β = 60°	0.7 05142	02527	$\bar{9}$9961	$\bar{9}$7446	$\bar{9}$4985	$\bar{9}$2584	$\bar{9}$0244	$\bar{8}$7972
	α = 68°	69°	70°	71°	72°	73°	74°	75°

The function $\Lambda_0(\alpha, \beta)$

	α = 60°	61°	62°	63°	64°	65°	66°	67°
β = 60°	0.7 27455	24554	21680	18836	16024	13246	10504	07802
61	37360	34513	31694	28903	26144	23419	20731	18081
62	47190	44401	41639	38906	36204	33536	30904	28310
63	56945	54217	51516	48845	46204	43596	41024	38490
64	66627	63964	61329	58721	56145	53601	51092	48621
65°	0.7 76237	73644	71077	68538	66029	63552	61110	58704
66	85778	83257	80762	78295	75857	73451	71079	68742
67	95251	92806	90387	87994	85631	83299	81000	78735
68	0.8 04658	02292	$\bar{9}9952$	$\bar{9}7638$	$\bar{9}5353$	$\bar{9}3097$	$\bar{9}0874$	$\bar{8}8685$
69	14001	11718	09460	07228	05023	02848	00704	$\bar{9}8593$
70°	0.8 23283	21086	18913	16765	14645	12552	10490	08460
71	32504	30397	28312	26252	24219	22212	20235	18289
72	41669	39654	37661	35691	33747	31829	29940	28080
73	50780	48859	46960	45084	43232	41406	39606	37835
74	59837	58015	56213	54433	52676	50943	49236	47556
75°	0.8 68846	67124	65421	63740	62080	60443	58831	57245
76	77807	76188	74588	73007	71447	69909	68394	66903
77	86724	85211	83715	82237	80779	79342	77926	76533
78	95600	94194	92805	91432	90078	88744	87429	86135
79	0.9 04438	03142	01860	00595	$\bar{9}9347$	$\bar{9}8117$	$\bar{9}6905$	$\bar{9}5713$
80°	0.9 13240	12055	10884	09728	08588	07464	06357	05268
81	22010	20938	19880	18834	17803	16787	15786	14802
82	30750	29794	28849	27916	26995	26088	25195	24317
83	39464	38624	37794	36975	36166	35370	34586	33814
84	48155	47433	46719	46014	45320	44635	43960	43297
85°	0.9 56826	56223	55626	55038	54457	53885	53321	52767
86	65481	64997	64519	64047	63581	63123	62671	62227
87	74122	73759	73400	73045	72695	72351	72011	71678
88	82754	82511	82271	82035	81801	81571	81345	81122
89	91379	91257	91137	91019	90902	90787	90674	90562
β = 90°	1.0 00000	00000	00000	00000	00000	00000	00000	00000
	α = 60°	61°	62°	63°	64°	65°	66°	67°

The function $\Lambda_0(\alpha, \beta)$

	α = 68°	69°	70°	71°	72°	73°	74°	75°
β = 60°	0.7 05142	02527	$\bar{9}9961$	$\bar{9}7446$	$\bar{9}4985$	$\bar{9}2584$	$\bar{9}0244$	$\bar{8}7972$
61	15473	12910	10394	07929	05518	03164	00872	$\bar{9}8645$
62	25758	23249	20787	18375	16016	13714	11471	09293
63	35996	33545	31140	28784	26481	24232	22043	19916
64	46189	43799	41455	39158	36912	34721	32587	30514
65°	0.7 56338	54012	51731	49497	47312	45180	43104	41089
66	66444	64186	61971	59801	57680	55611	53596	51639
67	76508	74320	72174	70073	68018	66014	64062	62168
68	86532	84417	82343	80312	78327	76390	74505	72674
69	96517	94478	92478	90521	88607	86740	84923	83159
70°	0.8 06464	04504	02581	00699	$\bar{9}8860$	$\bar{9}7065$	$\bar{9}5319$	$\bar{9}3624$
71	16375	14496	12653	10849	09086	07367	05693	04069
72	26251	24455	22695	20971	19287	17645	16047	14495
73	36094	34384	32708	31067	29464	27901	26380	24903
74	45905	44283	42694	41138	39619	38136	36694	35294
75°	0.8 55685	54155	52654	51186	49751	48352	46990	45669
76	65438	64000	62590	61210	59862	58548	57270	56029
77	75164	73820	72503	71214	69955	68727	67533	66374
78	84864	83617	82394	81198	80029	78890	77781	76705
79	94542	93392	92266	91163	90087	89037	88015	87024
80°	0.9 04198	03148	02119	01112	00129	$\bar{9}9170$	$\bar{9}8237$	$\bar{9}7332$
81	13835	12885	11955	11045	10156	09290	08447	07629
82	23453	22606	21777	20965	20171	19398	18646	17917
83	33057	32313	31584	30871	30175	29496	28836	28196
84	42646	42007	41380	40767	40169	39585	39018	38467
85°	0.9 52223	51689	51166	50654	50154	49667	49193	48733
86	61790	61362	60943	60533	60132	59741	59361	58992
87	71350	71028	70713	70405	70104	69810	69525	69248
88	80903	80689	80478	80273	80072	79876	79685	79500
89	90453	90345	90240	90137	90037	89939	89843	89751
β = 90°	1.0 00000	00000	00000	00000	00000	00000	00000	00000
	α = 68°	69°	70°	71°	72°	73°	74°	75°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 75°$	76°	77°	78°	79°	80°	81°	82°
$\beta = 0°$	0.0 00000	00000	00000	00000	00000	00000	00000	00000
1	11960	11873	11789	11709	11631	11557	11486	11420
2	23919	23746	23578	23416	23261	23113	22972	22840
3	35876	35616	35365	35123	34890	34668	34458	34260
4	47829	47484	47150	46827	46517	46222	45941	45678
5°	0.0 59779	59348	58931	58528	58142	57773	57423	57095
6	71724	71208	70708	70226	69763	69322	68903	68509
7	83663	83062	82481	81920	81381	80867	80380	79922
8	95594	94910	94247	93608	92995	92409	91854	91332
9	0.1 07518	06751	06008	05291	04603	03946	03324	02739
10°	19433	18583	17761	16967	16206	15479	14790	14143
11	31338	30407	29506	28637	27803	27007	26252	25543
12	43232	42221	41242	40299	39393	38528	37709	36939
13	55114	54024	52969	51952	50976	50044	49161	48331
14	66983	65816	64686	63596	62551	61553	60607	59718
15°	0.1 78839	77596	76392	75231	74117	73054	72046	71099
16	90680	89362	88086	86856	85675	84548	83480	82476
17	0.2 02506	01115	$\bar{9}9768$	$\bar{9}8469$	$\bar{9}7223$	$\bar{9}6033$	$\bar{9}4906$	$\bar{9}3847$
18	14316	12853	11437	10071	08761	07510	06325	05212
19	26108	24575	23091	21661	20288	18978	17737	16570
20°	0.2 37883	36282	34732	33238	31804	30436	29140	27922
21	49638	47971	46357	44802	43309	41885	40535	39267
22	61375	59643	57967	56352	54802	53323	51922	50605
23	78091	71297	69561	67887	66282	64750	63299	61935
24	84785	82932	81138	79408	77749	76167	74667	73258
25°	0.2 96459	94547	92697	90914	89203	87571	86025	84573
26	0.3 08109	06142	04238	02403	00643	$\bar{9}8964$	$\bar{9}7373$	$\bar{9}5879$
27	19737	17716	15761	13877	12069	10345	08711	07177
28	31341	29269	27265	25333	23480	21713	20039	18466
29	42920	40800	38749	36772	34876	33068	31355	29746
$\beta = 30°$	0.3 54475	52309	50213	48194	46257	44410	42661	41017
	$\alpha = 75°$	76°	77°	78°	79°	80°	81°	82°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 83°$	84°	85°	86°	87°	88°	89°	90°
$\beta =$ 0°	0.0 00000	00000	00000	00000	00000	00000	00000	00000
1	11359	11303	11252	11207	11170	11140	11119	11111
2	22718	22605	22503	22414	22339	22280	22239	22222
3	34076	33907	33755	33621	33508	33419	33358	33333
4	45433	45208	45005	44828	44677	44559	44478	44444
5°	0.0 56789	56508	56256	56034	55846	55698	55597	55556
6	68143	67807	67505	67239	67015	66838	66716	66667
7	79496	79105	78753	78444	78183	77977	77836	77778
8	90847	90401	90000	89648	89350	89116	88955	88889
9	0.1 02195	01696	01246	00851	00517	00254	00074	00000
10°	0.1 13540	12988	12490	12053	11684	11392	11193	11111
11	24883	24278	23732	23254	22849	22530	22312	22222
12	36223	35565	34973	34453	34014	33668	33431	33333
13	47558	46850	46212	45651	45178	44805	44549	44444
14	58891	58132	57448	56848	56341	55942	55668	55556
15°	0.1 70219	69410	68682	68043	67504	67078	66786	66667
16	81542	80686	79914	79237	78665	78214	77905	77778
17	92862	91958	91143	90428	89825	89349	89023	88889
18	0.2 04176	03226	02370	01618	00984	00483	00141	00000
19	15485	14490	13593	12806	12142	11618	11258	11111
20°	0.2 26789	25750	24814	23992	23298	22751	22376	22222
21	38088	37006	36031	35176	34453	33884	33493	33333
22	49380	48257	47245	46357	45607	45016	44611	44444
23	60667	59504	58456	57536	56760	56147	55728	55556
24	71948	70746	69663	68713	67911	67278	66844	66667
25°	0.2 83222	81983	80867	79887	79060	78408	77961	77778
26	94489	93215	92066	91059	90208	89537	89077	88889
27	0.3 05750	04441	03262	02228	01354	00665	00194	00000
28	17004	15662	14454	13394	12499	11793	11309	11111
29	28250	26878	25642	24557	23642	22920	22425	22222
$\beta =$ 30°	0.3 39489	38088	36826	35718	34783	34046	33541	33333
	$\alpha = 83°$	84°	85°	86°	87°	88°	89°	90°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 75°$	76°	77°	78°	79°	80°	81°	82°
$\beta = 30°$	0.3 54475	52309	50213	48194	46257	44410	42661	41017
31	66004	63795	61657	59598	57623	55739	53955	52279
32	77508	75257	73081	70983	68972	67054	65238	63531
33	88984	86696	84483	82350	80305	78355	76509	74774
34	0.4 00434	$\bar{9}8111$	$\bar{9}5863$	$\bar{9}3698$	$\bar{9}1622$	$\bar{8}9642$	$\bar{8}7768$	$\bar{8}6007$
35°	0.4 11857	09500	07222	05026	02922	00915	$\bar{9}9014$	$\bar{9}7229$
36	23251	20865	18558	16335	14204	12173	10249	08442
37	34618	32205	29872	27624	25470	23416	21471	19644
38	45956	43519	41163	38893	36718	34644	32680	30836
39	57266	54808	52431	50142	47948	45857	43877	42018
40°	0.4 68546	66070	63676	61371	59161	57055	55061	53189
41	79797	77306	74898	72579	70356	68238	66232	64349
42	91019	88516	86096	83766	81533	79405	77390	75499
43	0.5 02211	$\bar{9}9699$	$\bar{9}7270$	$\bar{9}4932$	$\bar{9}2692$	$\bar{9}0556$	$\bar{8}8535$	$\bar{8}6637$
44	13373	10855	08421	06078	03832	01692	$\bar{9}9667$	$\bar{9}7765$
45°	0.5 24506	21985	19548	17202	14955	12813	10786	08883
46	35609	33087	30651	28306	26059	23918	21891	19989
47	46682	44164	41730	39388	37145	35007	32984	31085
48	57725	55213	52786	50450	48212	46081	44063	42169
49	68738	66235	63818	61491	59262	57139	55129	53243
50°	0.5 79721	77231	74826	72511	70294	68181	66183	64307
51	90675	88201	85810	83510	81307	79209	77223	75359
52	0.6 01600	$\bar{9}9144$	$\bar{9}6772$	$\bar{9}4489$	$\bar{9}2303$	$\bar{9}0220$	$\bar{8}8250$	$\bar{8}6401$
53	12496	10061	07710	05447	03281	01217	$\bar{9}9265$	$\bar{9}7433$
54	23362	20952	18625	16385	14241	12199	10267	08454
55°	0.6 34200	31818	29517	27303	25184	23166	21256	19464
56	45010	42658	40386	38202	36110	34118	32233	30465
57	55791	53472	51234	49080	47018	45055	43198	41455
58	66545	64263	62059	59939	57910	55978	54151	52436
59	77272	75028	72863	70780	68786	66887	65091	63407
$\beta = 60°$	0.6 87972	85770	83645	81601	79645	77782	76021	74368
	$\alpha = 75°$	76°	77°	78°	79°	80°	81°	82°

The function $\Lambda_0(\alpha, \beta)$

	α = 83°	84°	85°	86°	87°	88°	89°	90°
β = 30°	0.3 39489	38088	36826	35718	34783	34046	33541	33333
31	50721	49292	48005	46876	45923	45171	44656	44444
32	61945	60490	59180	58030	57060	56295	55771	55556
33	73162	71683	70351	69182	68196	67418	66885	66667
34	84370	82869	81517	80331	79330	78540	78000	77778
35°	0.3 95570	94049	92679	91477	90462	89662	89114	88889
36	0.4 06763	05223	03836	02619	01592	00782	00228	00000
37	17947	16390	14988	13758	12721	11902	11341	11111
38	29122	27551	26136	24894	23847	23020	22454	22222
39	40290	38705	37279	36027	34971	34138	33567	33333
40°	0.4 51449	49853	48417	47157	46093	45255	44680	44444
41	62599	60995	59550	58283	57214	56370	55792	55556
42	73741	72129	70679	69406	68332	67485	66905	66667
43	84874	83258	81802	80525	79448	78598	78016	77778
44	95999	94379	92921	91641	90562	89711	89128	88889
45°	0.5 07114	05494	04034	02754	01674	00823	00239	00000
46	18222	16602	15143	13864	12785	11933	11350	11111
47	29320	27703	26247	24970	23893	23043	22461	22222
48	40410	38798	37346	36073	34999	34151	33571	33333
49	51491	49886	48440	47172	46103	45259	44681	44444
50°	0.5 62564	60967	59529	58268	57205	56366	55791	55556
51	73628	72042	70614	69361	68305	67471	66901	66667
52	84684	83110	81693	80451	79403	78576	78010	77778
53	95731	94172	92768	91537	90499	89680	89119	88889
54	0.6 06770	05227	03838	02620	01593	00782	00228	00000
55°	0.6 17800	16276	14903	13700	12685	11884	11336	11111
56	28823	27318	25964	24776	23775	22985	22444	22222
57	39837	38355	37020	35850	34863	34085	33552	33333
58	50844	49385	48072	46921	45950	45184	44659	44444
59	61843	60409	59119	57988	57034	56282	55767	55556
β = 60°	0.6 72833	71427	70162	69053	68117	67379	66874	66667
	α = 83°	84°	85°	86°	87°	88°	89°	90°

The function $\Lambda_0(\alpha, \beta)$

	$\alpha = 75°$	76°	77°	78°	79°	80°	81°	82°
$\beta = 60°$	0.6 87972	85770	83645	81601	79645	77782	76021	74368
61	98645	96489	94407	92405	90488	88664	86938	85320
62	0.7 09293	07184	05148	03190	01316	$\bar{9}9532$	$\bar{9}7845$	$\bar{9}6262$
63	19916	17857	15869	13958	12128	10387	08740	07196
64	30514	28507	26571	24708	22926	21229	19625	18120
65°	0.7 41089	39137	37253	35442	33709	32059	30500	29036
66	51639	49745	47917	46160	44478	42877	41364	39944
67	62168	60333	58563	56862	55233	53683	52218	50844
68	72674	70902	69192	67548	65975	64479	63063	61736
69	83159	81451	79804	78220	76704	75262	73899	72620
70°	0.7 93624	91983	90399	88877	87421	86036	84726	83497
71	0.8 04069	02496	00979	$\bar{9}9521$	$\bar{9}8126$	$\bar{9}6799$	$\bar{9}5544$	$\bar{9}4367$
72	14495	12993	11544	10152	08820	07552	06354	05231
73	24903	23474	22095	20770	19503	18297	17157	16087
74	35294	33939	32633	31377	30175	29032	27951	26938
75°	0.8 45669	44390	43157	41972	40838	39759	38739	37783
76	56029	54828	53669	52556	51491	50478	49521	48623
77	66374	65252	64170	63130	62136	61190	60296	59457
78	76705	75664	74660	73695	72773	71895	71065	70287
79	87024	86065	85140	84252	83402	82593	81829	81112
80°	0.8 97332	96456	95612	94800	94024	93286	92588	91933
81	0.9 07629	06838	06074	05341	04640	03973	03342	02751
82	17917	17211	16530	15875	15250	14655	14092	13565
83	28196	27576	26978	26404	25855	25333	24839	24376
84	38467	37935	37421	36927	36455	36006	35582	35184
85°	0.9 48733	48288	47859	47446	47052	46677	46323	45990
86	58992	58636	58292	57962	57646	57345	57061	56795
87	69248	68980	68722	68474	68236	68011	67797	67597
88	79500	79322	79149	78984	78825	78675	78532	78399
89	89751	89661	89575	89492	89413	89338	89266	89200
$\beta = 90°$	1.0 00000	00000	00000	00000	00000	00000	00000	00000
	$\alpha = 75°$	76°	77°	78°	79°	80°	81°	82°

The function $\Lambda_0(\alpha, \beta)$

	α = 83°	84°	85°	86°	87°	88°	89°	90°
β = 60°	0.6 72833	71427	70162	69053	68117	67379	66874	66667
61	83817	82440	81201	80114	79198	78475	77981	77778
62	94793	93447	92235	91173	90277	89571	89087	88889
63	0.7 05762	04448	03266	02229	01355	00666	00194	00000
64	16723	15444	14292	13282	12431	11759	11300	11111
65°	0.7 27678	26434	25315	24333	23505	22852	22405	22222
66	38626	37419	36333	35381	34578	33945	33511	33333
67	49568	48400	47348	46426	45649	45036	44617	44444
68	60504	59375	58360	57470	56719	56127	55722	55556
69	71433	70346	69368	68510	67787	67217	66827	66667
70°	0.7 82357	81312	80373	79549	78854	78307	77932	77778
71	93275	92274	91375	90585	89920	89395	89036	88889
72	0.8 04188	03232	02373	01620	00984	00484	00141	00000
73	15095	14186	13369	12652	12048	11571	11245	11111
74	25998	25137	24362	23682	23110	22658	22349	22222
75°	0.8 36896	36083	35352	34711	34171	33745	33453	33333
76	47790	47026	46340	45738	45231	44831	44557	44444
77	58679	57967	57326	56764	56290	55916	55660	55556
78	69565	68904	68309	67788	67348	67001	66764	66667
79	80447	79838	79290	78810	78405	78086	77867	77778
80°	0.8 91326	90770	90270	89831	89462	89170	88971	88889
81	0.9 02202	01700	01248	00852	00518	00254	00074	00000
82	13075	12627	12224	11871	11573	11338	11177	11111
83	23946	23553	23199	22889	22627	22421	22280	22222
84	34815	34477	34173	33906	33682	33504	33383	33333
85°	0.9 45682	45400	45145	44923	44735	44587	44486	44444
86	56547	56321	56117	55939	55789	55670	55589	55556
87	67412	67242	67089	66955	66842	66753	66692	66667
88	78275	78161	78059	77970	77895	77835	77794	77778
89	89138	89081	89030	88985	88947	88918	88897	88889
β = 90°	1.0 00000	00000	00000	00000	00000	00000	00000	00000
	α = 83°	84°	85°	86°	87°	88°	89°	90°

The function $A_0(\alpha, \beta)$

	$\alpha =$	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°
$\beta = 80°$	0.98	4808	4807	4805	4801	4796	4789	4781	4771	4760	4747
81		7688	7687	7685	7681	7676	7669	7661	7651	7640	7627
82	0.99	0268	0267	0265	0261	0256	0249	0241	0231	0220	0207
83		2546	2545	2543	2539	2534	2527	2519	2509	2498	2485
84		4522	4521	4519	4516	4510	4504	4495	4485	4474	4461
85	0.99	6195	6194	6192	6188	6183	6176	6168	6158	6147	6134
86		7564	7563	7561	7558	7552	7545	7537	7527	7516	7503
87		8630	8629	8627	8623	8618	8611	8602	8592	8581	8569
88		9391	9390	9388	9384	9378	9372	9364	9354	9344	9332
$\beta = 89°$		9848	9846	9843	9838	9833	9827	9820	9812	9804	9795

	$\alpha =$	2.0°	2.1°	2.2°	2.3°	2.4°	2.5°	2.6°	2.7°	2.8°	2.9°
$\beta = 80°$	0.98	4511	4481	4449	4416	4382	4346	4309	4270	4230	4189
81		7391	7361	7329	7296	7262	7227	7190	7152	7112	7071
82		9971	9941	9910	9877	9843	9808	9771	9733	9694	9654
83	0.99	2250	2220	2189	2156	2123	2088	2051	2014	1975	1935
84		4227	4198	4167	4135	4102	4067	4032	3995	3957	3918
85	0.99	5903	5874	5844	5813	5780	5747	5712	5677	5640	5603
86		7277	7249	7221	7191	7160	7129	7096	7063	7028	6993
87		8353	8327	8300	8273	8244	8215	8186	8156	8125	8094
88		9139	9117	9094	9071	9048	9025	9001	8977	8952	8927
$\beta = 89°$		9660	9646	9633	9619	9605	9591	9576	9562	9548	9533

	$\alpha =$	4.0°	4.1°	4.2°	4.3°	4.4°	4.5°	4.6°	4.7°	4.8°	4.9°
$\beta = 80°$	0.98	3652	3596	3538	3479	3420	3359	3297	3233	3169	3104
81		6539	6483	6427	6369	6310	6250	6189	6127	6063	5999
82		9129	9075	9019	8962	8905	8846	8786	8726	8664	8601
83	0.99	1422	1369	1315	1260	1205	1148	1090	1032	0972	0912
84		3421	3370	3319	3266	3213	3159	3104	3048	2992	2935
85	0.99	5130	5082	5034	4985	4935	4885	4834	4782	4730	4677
86		6557	6514	6470	6426	6381	6336	6290	6244	6198	6151
87		7717	7680	7643	7606	7568	7530	7492	7454	7416	7377
88		8638	8611	8584	8556	8529	8501	8473	8445	8417	8389
$\beta = 89°$		9372	9357	9343	9328	9313	9298	9284	9269	9254	9239

The function $\Lambda_0(\alpha, \beta)$

	$\alpha =$	1.0°	1.1°	1.2°	1.3°	1.4°	1.5°	1.6°	1.7°	1.8°	1.9°
$\beta =$ 80°	0.98	4733	4717	4700	4682	4662	4640	4617	4593	4567	4540
81		7613	7597	7580	7562	7542	7520	7497	7473	7447	7420
82	0.99	0193	0177	0160	0141	0121	0100	0077	0053	0027	0000
83		2471	2456	2439	2420	2400	2379	2356	2331	2305	2278
84		4447	4431	4414	4396	4376	4354	4332	4308	4282	4255
85	0.99	6120	6105	6088	6069	6049	6028	6006	5982	5957	5931
86		7489	7473	7456	7438	7419	7398	7376	7353	7329	7304
87		8555	8540	8523	8506	8487	8467	8446	8425	8402	8378
88		9319	9304	9289	9273	9256	9238	9220	9200	9180	9160
$\beta =$ 89°		9785	9774	9762	9750	9738	9725	9713	9700	9687	9673

	$\alpha =$	3.0°	3.1°	3.2°	3.3°	3.4°	3.5°	3.6°	3.7°	3.8°	3.9°
$\beta =$ 80°	0.98	4147	4103	4058	4012	3964	3915	3865	3814	3761	3707
81		7029	6986	6941	6895	6848	6799	6750	6699	6647	6593
82		9612	9569	9525	9479	9433	9385	9336	9286	9235	9182
83	0.99	1894	1852	1808	1764	1718	1671	1624	1575	1525	1474
84		3878	3837	3794	3751	3707	3661	3615	3568	3520	3471
85	0.99	5564	5524	5484	5442	5400	5357	5313	5268	5223	5177
86		6957	6920	6882	6843	6804	6764	6724	6683	6642	6600
87		8062	8029	7996	7963	7929	7895	7860	7825	7789	7753
88		8902	8876	8850	8824	8798	8772	8745	8719	8692	8665
$\beta =$ 89°		9519	9504	9490	9475	9460	9446	9431	9416	9401	9387

	$\alpha =$	5.0°	5.1°	5.2°	5.3°	5.4°	5.5°	5.6°	5.7°	5.8°	5.9°
$\beta =$ 80°	0.98	3037	2969	2901	2831	2760	2689	2616	2542	2467	2392
81		5934	5868	5801	5732	5663	5593	5522	5450	5377	5304
82		8538	8474	8408	8342	8275	8207	8138	8069	7998	7927
83	0.99	0851	0789	0726	0662	0598	0533	0467	0400	0333	0265
84		2877	2818	2759	2699	2638	2577	2515	2453	2390	2327
85	0.99	4624	4570	4515	4460	4405	4349	4293	4236	4179	4121
86		6104	6056	6008	5960	5911	5862	5813	5763	5713	5663
87		7338	7299	7259	7220	7180	7140	7100	7060	7019	6979
88		8361	8333	8304	8276	8247	8219	8190	8161	8133	8104
$\beta =$ 89°		9224	9209	9194	9179	9164	9149	9134	9119	9104	9089

Appendix

As examples of the use of the tables we shall here discuss some dynamical problems, where the required quantities are expressible in terms of the functions $F_0(\alpha)$, $E_0(\alpha)$ and $\Lambda_0(\alpha, \beta)$, and thus can be computed numerically with the aid of the present tables.

I. The spherical pendulum

Let O be the point of suspension and a the length of the string. The bob A of the pendulum is then constrained to move on the surface of a sphere with centre in O and of radius $OA = a$. Let the lowest point of this sphere be Z, the highest Z'. We consider the sphere as a surface of revolution, whose meridians are the semicircles on ZZ' as diameter and whose parallels are horizontal circles.

In the general type of motion the meridian through the bob turns steadily in the same direction about the vertical ZZ'. The rotational angle, counted from the initial position, is called the *azimuth* and may be denoted by ψ. Further we denote by θ the angular distance ZOA from the lowest point Z to the bob A. The instantaneous position of the bob is then determined by these two angles, θ and ψ.

While the azimuth ψ steadily increases, the angle θ varies between two constant limits, $\theta_1 > 0$ and $\theta_2 > \theta_1$. The bob then describes a twisted curve on the spherical zone which is bounded by the parallels corresponding to $\theta = \theta_1$ and $\theta = \theta_2$. The path touches alternately these limiting parallels. The points of tangency form the apsides of the path and divide it into congruent arcs. Between two consecutive apsides the azimuth ψ increases by a constant amount, which may be called the *apsidal angle* and denoted by Ψ. The individual arcs between consecutive apsides are described in equal times. We shall denote by T the time it takes for the bob to move through two such arcs, i.e., from one apsis to the next one on the same level. Thus T is the *period* for the angle θ, if expressed as a function of the time.

In such a "zonal" motion of the pendulum, θ_1 is always $< \pi/2$, whereas θ_2 may also be $> \pi/2$. Thus the lower limiting parallel lies always below the point O, while the upper one may however lie above this level. There can then be a possibility that the string slackens, but we shall suppose this possibility eliminated by the convention that in such a case the string is replaced by a weightless rod.

There is also another condition for the angles θ_1 and θ_2, viz. that $\theta_1 + \theta_2 < \pi$, so that $0 < \theta_1 < \theta_2 < \pi - \theta_1$.

The period T and the apsidal angle Ψ can now be expressed by the functions $F_0(\alpha)$ and $\Lambda_0(\alpha, \beta)$, as follows.

Instead of the angles θ_1, θ_2 we introduce their cosines putting

$$z_1 = \cos \theta_1, \qquad z_2 = \cos \theta_2.$$

Here z_1 is then always >0, whereas z_2 can be $>$, $=$ or <0. Further we have $z_1 + z_2 > 0$, so that $1 > z_1 > z_2 > -z_1$.

Let s_1, s_2, q_1, q_2 be the positive square roots of

$$s_1^2 = 1 - z_1^2, \qquad q_1^2 = 1 + 2z_1z_2 + z_1^2,$$
$$s_2^2 = 1 - z_2^2, \qquad q_2^2 = 1 + 2z_1z_2 + z_2^2.$$

We then define the two angles α and β by the formulas

$$\cos \alpha = \frac{q_2}{q_1}, \qquad 0 < \alpha < \pi/2, \qquad \cos \beta = \frac{s_1 z_2}{q_2}, \qquad 0 < \beta < \pi.$$

If $z_2 < 0$, then $\pi/2 < \beta < \pi$, and we have then, according to §8, $\Lambda_0(\alpha, \beta) = 2 - \Lambda_0(\alpha, \pi - \beta)$.

The period T has the value

$$T = F_0(\alpha) \cdot \pi \sqrt{\frac{l}{g}}, \quad \text{where} \quad l = \frac{z_1 + z_2}{q_1^2} \cdot 2a,$$

and the apsidal angle Ψ is determined by the expression

$$\Psi = \left\{ \frac{s_1 s_2}{q_1} F_0(\alpha) + \Lambda_0(\alpha, \beta) \right\} \cdot \frac{\pi}{2}.$$

We shall here consider particularly the latter function. With the aid of the tables of $F_0(\alpha)$ and $\Lambda_0(\alpha, \beta)$ it has been possible to compute Ψ for a great number of chosen values of the two variables z_1, z_2, and in this way investigate, how Ψ varies with them. The result is most easily depicted by a diagram. We interpret z_1 and z_2 as Cartesian coördinates of the points in a plane and in this plane we plot curves $\Psi = \textit{constant}$, for a suitable sequence of values of the constant. In consequence of the conditions on z_1 and z_2, mentioned above, the points corresponding to zonal motion must lie in the interior of the isosceles triangle bounded by the lines $z_1 = 1$, $z_1 = z_2$ and $z_1 = -z_2$.

Fig. 3 shows such a diagram, in which several Ψ-curves have been plotted. The positive direction of the z_1-axis is vertically downwards, the positive direction of the z_2-axis is horizontally to the right, and the origin is at the top vertex of the triangle.

For the motion in question Puiseux and Halphen have shown, that $90° < \Psi < 180°$, and this fact is also confirmed by the diagram.

For given values of z_1 and z_2 one can obtain an approximate value of the corresponding apsidal angle from this diagram. Thus for $z_1 = 0.8$, $z_2 = 0.6$ we find $\Psi \sim 113°$. More accurate computation gives 113.21.°

While every point in the interior of the triangle represents a zonal motion, the points on the sides correspond to certain limiting cases, in which the path is a plane curve, hence a circle or a circular arc. A point on the line $z_1 = 1$ represents oscillations in a vertical plane, in which θ varies between $\theta_1 = 0$ and θ_2 (simple pendulum). A point on the line $z_1 = z_2$ represents rotation in a horizontal circle, in which θ is

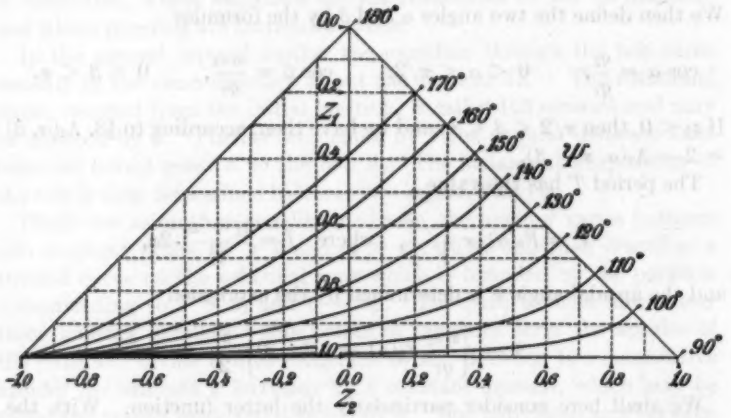

FIG. 3

constant (conical pendulum). The case $z_1 = -z_2$ corresponds to motion in a great circle, but, if $\theta_1 \neq 0$, this motion cannot be realized physically, since it implies infinitely large velocity. If the velocity of the bob is very large, the tension in the string or the rod will also be very large and the influence of gravity on the motion will be relatively small. If this influence can be completely neglected, the path becomes a geodesic upon the sphere, i.e., a great circle, and hence $\theta_2 = \pi - \theta_1$ and $z_2 = -z_1$.

In the above formula for the apsidal angle, $z_1 = 1$ gives $\Psi = \pi/2$ and $z_1 = -z_2$ gives $\Psi = \pi$. These results have obvious interpretations in the corresponding motions. It follows that the base line $z_1 = 1$

and the left side $z_1 = -z_2$ are themselves members of the curve system Ψ = constant. On the other hand, $z_1 = z_2$ gives the variable value $\Psi = (1 + 3z_1^2)^{-\frac{1}{2}} \cdot \pi$. Since the path in this case is a horizontal circle, there are no apsides. However, if we consider this motion as a limiting case of a zonal motion, it is easily shown, that the apsidal angle tends to this particular value, when the upper limiting circle tends to coincide with the lower.

The diagram can also be used for the purpose of deriving approximate formulas for Ψ, which, however, must be omitted here.

(Cf. C. Heuman, Bidrag till teorien för sferiska pendeln, in "Skrifter utgivna av Tekniska Högskolan," Stockholm 1918.)

II. Motion on certain surfaces

The problem of the motion of the spherical pendulum is a special case of a more general problem, that of the motion of a particle on a smooth surface of revolution with vertical axis, under the influence of gravity and the reaction of the surface only. Under certain assumptions this motion becomes completely analogous to that of the spherical pendulum. The particle will move on a zone between two fixed parallels, which the path touches alternately. The points of tangency are apsides of the path and divide it into congruent arcs, which are described in equal times. The period T and the apsidal angle Ψ can then be defined as in the case of the spherical pendulum.

Now there are certain special surfaces on which T and Ψ can be expressed in terms of complete elliptic integrals. The integrals of the third kind which appear in these cases belong always to the circular class and are consequently expressible in terms of $\Lambda_0(\alpha, \beta)$.

The simplest of these surfaces are, next to the sphere, the cone and the paraboloid of revolution, and we shall here give the corresponding formulas.

We consider first the motion on a cone with vertical axis and denote by ϵ the angle which the generatrices form with the axis. Zonal motion can occur only on that sheet which is located above the vertex. If the limiting parallels are at the heights h and λh above the vertex, where $0 < \lambda < 1$, then

$$T = \frac{\pi}{\cos\epsilon}\left[\frac{1+2\lambda}{1+\lambda}\cdot\frac{2h}{g}\right]^{\frac{1}{2}}\cdot\left\{E_0(\alpha) - \frac{\lambda}{1+2\lambda}F_0(\alpha)\right\}.$$

$$\Psi = \frac{\pi}{\sin\epsilon}\Lambda_0(\alpha, \beta),$$

α and β being angles in the interval $(0, \pi/2)$ determined by

$$\sin^2\alpha = \frac{1-\lambda^2}{1+2\lambda}, \qquad \sin^2\beta = \frac{1}{2+\lambda}.$$

On a paraboloid of evolution with vertical axis zonal motion is possible only if the surface lies above the tangent plane at the vertex. If the focus is at the height a and the limiting parallels are at the heights μa and νa above this plane, where $\mu > \nu > 0$, then

$$T = \pi\left[(1+\mu)\cdot\frac{2a}{g}\right]^{\frac{1}{2}}\cdot E_0(\alpha),$$

$$\Psi = \left\{\left[\frac{\mu\nu}{\mu+\nu}\right]^{\frac{1}{2}} F_0(\alpha) + \Lambda_0(\alpha,\beta)\right\}\cdot\frac{\pi}{2}.$$

Here α and β are angles in the interval $(0, \pi/2)$ determined by

$$\sin^2\alpha = \frac{\mu-\nu}{1+\mu}, \qquad \sin^2\beta = \frac{1}{1+\nu}.$$

III. The gyroscopic pendulum

This pendulum consists of a spindle and a symmetric flywheel, which can rotate without friction about the spindle. One end of the spindle is suspended at a fixed point O in such a way that it is freely movable about O.

Let the other end-point of the spindle be S. If the spindle is released from any leaning position, with or without an initial velocity, while the wheel rotates about OS, the point S will describe a certain path, located on the sphere to which the point is bound. It is this path that we shall consider here in a certain special case.

In order to indicate the instantaneous position of the point S we use here the same angles, θ and ψ, as in the case of the spherical pendulum.

The centre of gravity of the wheel lies on the line OS, we suppose between O and S, at the distance a from O. Further we denote by m the mass of the wheel, by C its moment of inertia about the axis OS and by A its moment of inertia about an arbitrary axis through O and perpendicular to OS. It is assumed that the mass of the spindle can be neglected.

With different initial conditions the path of the point in question will present very different shapes. Here we restrict ourselves to considering the following case. We suppose that the axis OS at the first moment is at rest in such a position that $\theta = \theta_0$, while the wheel rotates

about OS with a given angular velocity ω_0. From this position the spindle is released without initial velocity.

The path of S then has the shape indicated in fig. 4. It is located between two parallel circles, of which the upper one passes through the initial position S_0. On this parallel the path has a sequence of cusps, $S_0, S_2, S_4, \cdots$, where the tangent is perpendicular to the circle. On the other hand, on the lower limiting parallel the path has a sequence of points, $S_1, S_3, \cdots$, where it is touched by this circle. The individual arcs $S_0S_1, S_1S_2, \cdots$, are all congruent to each other. If we denote by (θ_n, ψ_n) the angular coördinates of S_n, we have then $\theta_{n+2} = \theta_n$ and $\psi_{n+1} - \psi_n =$ constant. This constant is the *apsidal angle* Ψ. Further we put $\theta_0 - \theta_1 = \Theta$ and call Θ the *zonal angle*.

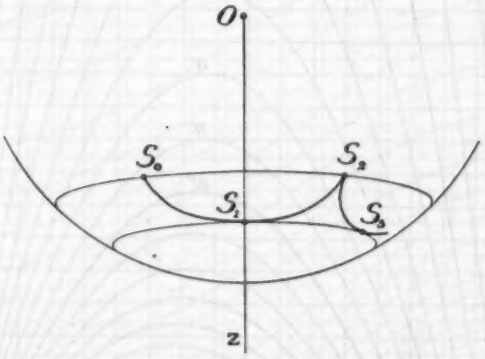

FIG. 4

The shape of the path is essentially characterized by the two angles Θ and Ψ. These in their turn are functions of two independent variables, θ_0 and μ. The former is, as already mentioned, the initial value of θ; the latter is the ratio

$$\mu = \frac{C^2\omega_0^2}{4Aamg},$$

which has the dimension zero. Here $0 < \theta_0 < \pi$ and $\mu > 0$. The number μ may be called the *coefficient of stability*.

If λ is the positive square root of

$$\lambda^2 = 1 + \mu^2 + 2\mu \cos\theta_0,$$

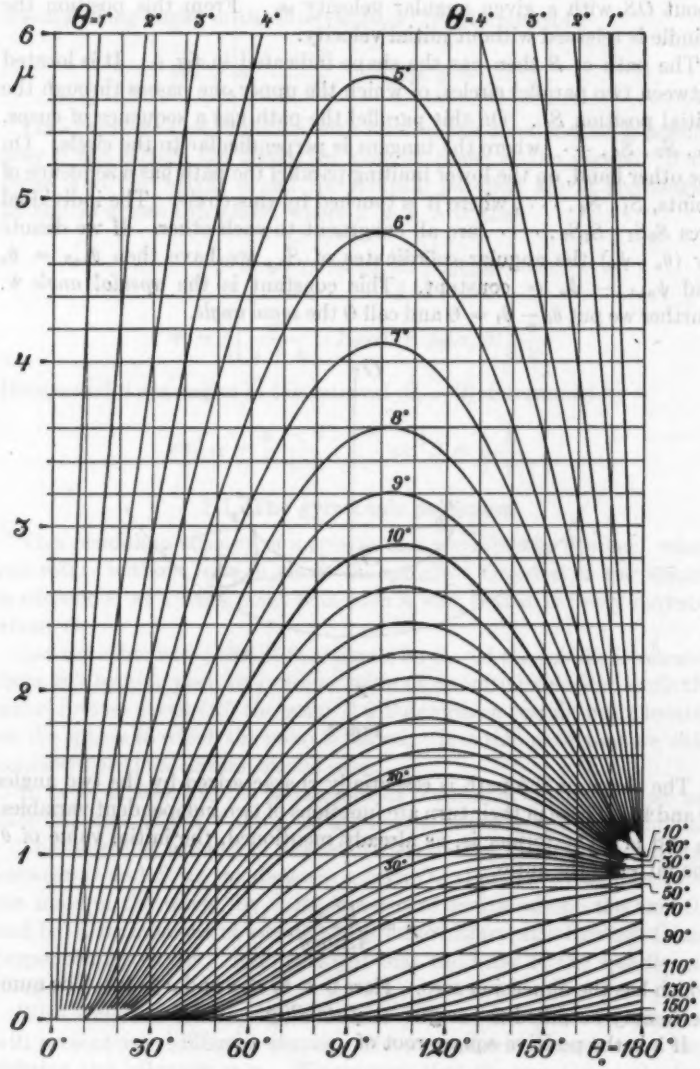
θ-1°
2°
3°
4°
θ-4°
3°
2°
1°
μ
6
5
4
3
2
1
0
5°
6°
7°
8°
9°
10°
20°
30°
10°
20°
30°
40°
50°
70°
90°
110°
130°
150°
170°
0°
30°
60°
90°
120°
150°
θ-180°

FIG. 5

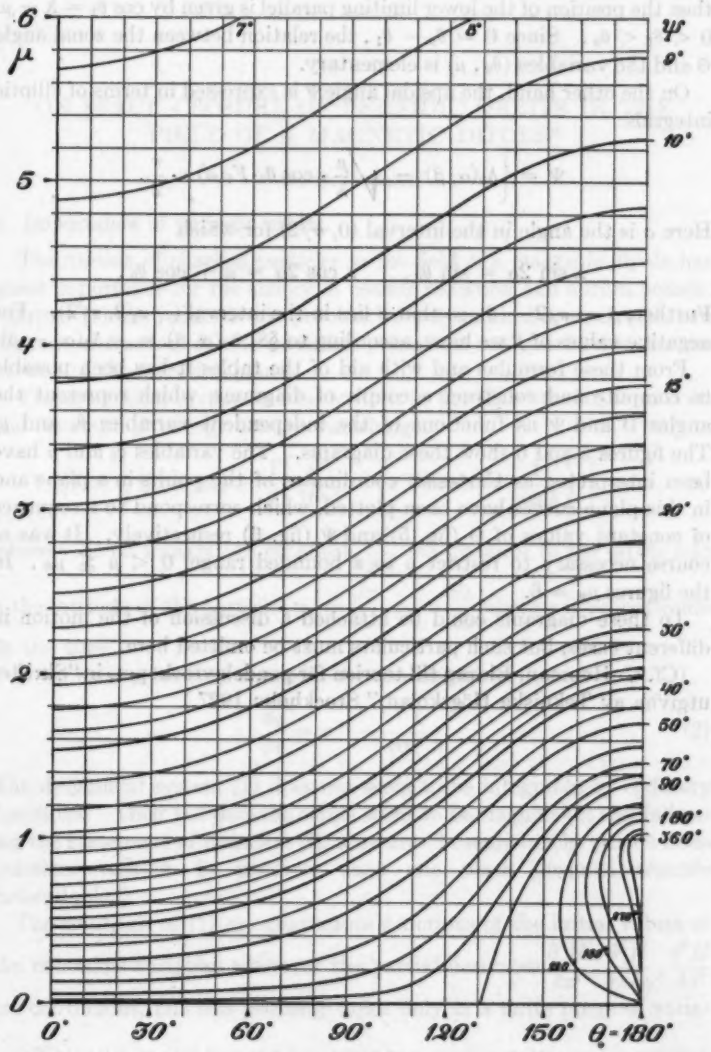
μ
ψ
6
5
4
3
2
1
0
7°
6°
0°
10°
15°
20°
30°
40°
50°
70°
90°
180°
360°
0°
30°
60°
90°
120°
150°
θ=180°

FIG. 6

then the position of the lower limiting parallel is given by $\cos \theta_1 = \lambda - \mu$, $0 < \theta_1 < \theta_0$. Since $\Theta = \theta_0 - \theta_1$, the relation between the zonal angle Θ and the variables (θ_0, μ) is elementary.

On the other hand, the apsidal angle Ψ is expressed in terms of elliptic integrals:

$$\Psi = \left\{ \Lambda_0(\alpha, \beta) - \sqrt{\frac{\mu}{\lambda}} \cdot \cos \theta_0 \cdot F_0(\alpha) \right\} \cdot \frac{\pi}{2}.$$

Here α is the angle in the interval $(0, \pi/2)$ for which

$$\lambda \sin 2\alpha = \sin \theta_0, \qquad \lambda \cos 2\alpha = \mu + \cos \theta_0.$$

Further, $\beta = \pi/2 - \theta_1$, so that β lies in the interval $(-\pi/2, \pi/2)$. For negative values of β we have, according to §8, $\Lambda_0(\alpha, \beta) = -\Lambda_0(\alpha, -\beta)$.

From these formulas and with aid of the tables it has been possible to compute and construct a couple of diagrams, which represent the angles Θ and Ψ as functions of the independent variables θ_0 and μ. The figures 5 and 6 show these diagrams. The variables θ_0 and μ have been interpreted as Cartesian coordinates of the points in a plane and in this plane curves have been plotted, which correspond to a sequence of constant values of Θ (fig. 5) and Ψ (fig. 6) respectively. It was of course necessary to restrict μ to a bounded range, $0 < \mu \leqq \mu_*$. In the figures $\mu_* = 6$.

To these diagrams could be attached a discussion of the motion in different cases, but such particulars must be omitted here.

(Cf. C. Heuman, Bidrag till teorien för pendelgyroskopet, in "Skrifter utgivna av Tekniska Högskolan," Stockholm 1927.)

ON SPACE CLOSURE OF PERIODIC ORBITS IN THE FIELD OF A MAGNETIC DIPOLE*

By O. Godart†

1. Importance of periodic orbits.

The motion of charged particles in the field of a magnetic dipole has great importance for the theory of cosmic radiation and aurora polaris. Störmer [1] has reduced this problem to a dynamical system with two degrees of freedom in the meridian plane and one quadrature for the motion of the meridian plane.

The dynamical system can be written in the Hamiltonian form with the constant of energy always vanishing:

$$H = \frac{1}{2}(p_x^2 + p_\lambda^2) - \frac{1}{2}\left[\frac{e^{2\sigma}}{16\gamma_1^4} - \left(e^{-\sigma}\cos\lambda - \frac{1}{\cos\lambda}\right)^2\right] = 0 \qquad (1)$$

where γ_1 is an integration constant, $\frac{e^\sigma}{2\gamma_1}$ the distance to the dipole and λ the latitude of the particle; $p_x = \frac{dx}{d\sigma}$, $p_\lambda = \frac{d\lambda}{d\sigma}$ where σ is proportional to the time.

If we write φ for the longitude, the quadrature is:

$$\frac{d\varphi}{d\sigma} = e^{-\sigma} - \frac{1}{\cos^2\lambda} \qquad (2)$$

The dynamical system (1) does not seem to be integrable by ordinary functions. Then the periodic orbits seem to be singular [2] and following the statement of Poincaré [3], they are "la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable."

The solutions of (1) are continuous functions of the initial values of the canonical variables wherever the partial derivates $\frac{\partial^2 H}{\partial x^2}, \frac{\partial^2 H}{\partial x\,\partial y}, \frac{\partial^2 H}{\partial x^2}$ are continuous, but this property holds only in a finite range of varia-

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tion of the functions and of the independent variable. Thus when we change infinitely slightly the initial conditions, the integrals will be modified by infinitesimal quantities. This is still valid for the periodic orbits, but these orbits are singular and the neighboring orbits are then no longer periodic. When the time increases indefinitely, the previous theorem will break down and the infinitely near orbit can be at infinitely great distance from the periodic orbit which remains at finite distance.

As a consequence, the periodic orbits will be a kind of discontinuity among the different bundles of same qualitative character. The orbits asymptotic to the periodic orbits remaining at finite distance when the time increases indefinitely will play the same role. The knowledge of the periodic and asymptotic orbits will give us the qualitative solution of the problem.

2. Analytical continuation.

If the Hamiltonian depends on a parameter, and if for a given value of this parameter, a point of equilibrium or a periodic trajectory is a solution of the equations, Poincaré [4] has proved that periodic orbits still exist in the infinitesimal neighborhood, under certain conditions. Birkhoff [5] proved it for a longer range of the parameter when the Hamiltonian is also function of the time. If the Hamiltonian is not a function of the time and has a point of equilibrium or a periodic solution, it is always possible to reduce the problem to dynamical system of one less degree of freedom, but with an Hamiltonian which is a periodic function of the time.

When the starting orbit is periodic, we can proceed as Birkhoff [6] but without assigning a particular value to the parameter. If we know a point of equilibrium where the parameter a is equal to a_0 and the canonical coördinates have the values p_i^0, q_i^0 ($i = 1, \cdots m$ where m is the number of degrees of freedom), we may change the variables

$$p_i = p_i^0 + p_i^1 \qquad q_i = q_i^0 + q_i^1 \tag{3}$$

If we neglect the square of p_i^1, q_i^1 we obtain linear differential equations; let us call Ω_i the coefficients of the exponential solutions of these equations. To have still periodic orbits, one among the Ω_i's, which we shall call Ω, must be imaginary. It is possible [7] by contact transformation to have among the canonical variables two variables p and q such that the vibrations around the point of equilibrium are expressed by $p = ce^{\Omega t}$, $q = de^{-\Omega t}$ where c and d are constant and t is the time.

Let us make the contact transformation:

$$p = \sqrt{\Omega i 2K}\sin\tau \qquad q = \sqrt{\frac{2K}{\Omega i}}\cos\tau \tag{4}$$

The Hamiltonian in the new variables K, τ is periodic in τ of period 2π. Solving $H = h$ for K, we can decrease by one, as for the periodic orbits [6], the number of degrees of freedom, taking K as new Hamiltonian and τ as new time.

Let us choose a dynamical system with one degree of freedom, noting that there is no essential difference if the system has more. The Hamiltonian K, periodic function of τ of period 2π and function of a parameter a, has for $a = a_0$ a point of equilibrium given by the canonical variables $P = 0$, $Q = 0$. The general solution can be written

$$Q = Q(P_0, Q_0, \tau, a) \qquad P = P(P_0, Q_0, \tau, a)$$

where P_0, Q_0 are the values of P, Q when $\tau = 0$. If the solution is periodic

$$Q(P_0, Q_0, 2\pi, a) = Q_0 \qquad P(P_0, Q_0, 2\pi, a) = P_0 \tag{5}$$

In order that P_0, Q_0 can be found to satisfy these relations Poincaré [4] and after him Birkhoff [5] proved that the variational equations near the point of equilibrium have for $a = a_0$ no characteristic exponent vanishing or equal to $\sqrt{-1}$. Moreover, Birkhoff [5] proved that the analytical continuation of the periodic orbits is possible until the value of a is reached for which the characteristic exponent vanishes.

If we eliminate between the two equations one variable, for example P_0, we obtain

$$F(Q_0, a) = 0 \tag{6}$$

To each value of a, there correspond in (6) a certain number of roots of Q_0, each of them being the initial value of a periodic orbit. When a varies, the number of roots changes but the roots disappear in pairs and therefore also the periodic orbits. When the characteristic exponent vanishes, we cannot continue any more our expansion; this means that for this value of the parameter at least two periodic orbits coincide before disappearing. For this value of a, a^*; the equation (6) has a multiple root.

Let us suppose that the root Q_0 is of the order of multiplicity n. Then following Poincaré [8] we can find convergent solutions of (6) called of index p

$$Q_0 - Q_0^* = \alpha_1\epsilon(a - a^*)^{1/p} + \alpha_2\epsilon^2(a - a^*)^{2/p} \cdots \qquad (p \text{ integer})$$

where ϵ is a p^{th} root of unity and the α's are constant. If we have q_p groups of this form, $\sum_p pq_p = n$.

In the case $n = 2$, we have the two expansions:

$$Q_0 - Q_0^* = \alpha_1(a - a^*)^{\frac{1}{2}} + \alpha_2(a - a^*) \cdots$$
$$Q_0 - Q_0^* = -\alpha_1(a - a^*)^{\frac{1}{2}} + \alpha_2(a - a^*) \cdots \quad (7)$$

All the other parameters of the periodic orbits will be expanded in the same way.

3. Forms of expansion of periodic and neighboring orbits.

Let us apply these theorems to our problem. The equations (1) admit for $\gamma_1 = 1$ a point of equilibrium $x = \log 2$, $\lambda = 0$. The vibration around this point of equilibrium has an imaginary Ω, $\frac{\sqrt{-3}}{2}$ and we shall have a family of periodic orbits for other values of γ_1 or $a = \frac{1}{16\gamma_1^4}$ until the characteristic exponent vanishes; for this value of γ_1, at least one other family of periodic orbits will coincide with the first one, and all the parameters of the periodic orbits can be expanded in power series of $(a - a^*)^{1/p}$.

Lemaître [9] has developed a method of calculating the periodic orbits with great precision, for a given γ_1, and actually computed two of them. The author [10] following Lemaître's method has computed some orbits of the first family and in particular the limiting one. Baños and his collaborators [11] using a method of numerical integration similar to the method previously devised by Störmer [12] have computed the orbits of the other branch for the same γ_1 we were concerned with.

The periodic orbits are expressed as Fourier's series:

$$x = x_0 + x_2 \cos 2\omega\sigma + x_4 \cos 4\omega\sigma + x_6 \cos 6\omega\sigma \cdots$$
$$\lambda = \lambda_1 \sin \omega\sigma + \lambda_3 \sin 3\omega\sigma + \lambda_5 \sin 5\omega\sigma$$

Lemaître found [9] for the first branch that λ can be expressed by

$$\sin \lambda = \sqrt{z_1} \sin [\omega\sigma + \psi_1 \sin 2\omega\sigma + \psi_2 \sin 4\omega\sigma \cdots] \quad (9)$$

and that ψ_2 was negligible to 6 decimals.

We found that the other branch can be expressed in the same way but ψ_2 is no longer negligible. We have then computed by harmonic analysis z_1, ψ_1, ψ_2 for the orbits studied by Baños and his collaborators.

To know the orbits in space, we must still compute the longitude φ by the quadrature (2); we write

$$\varphi - C = \varphi_0\omega\sigma + \varphi_2 \sin 2\omega\sigma + \varphi_4 \sin 4\omega\sigma \cdots \qquad (10)$$

where C is a constant of integration and φ_0, φ_2, $\varphi_4 \cdots$ constants. We have computed φ for all the orbits studied. All the coefficients of the periodic orbits are given in Tables I and II.

TABLE I (unstable)

η	.0	.01	.02	.03	.04	.05	.06	.067,551
γ_1	1	.964,953	.929,898	.895,050	.860,846	.828,245	.799,998	.788,541
μ	1	.913,378	.817,609	.709,708	.584,751	.433,313	.232,786	0
τ_0	1	.973,854	.945,009	.912,620	.875,311	.830,479	.772,002	.705,495
ω	.866,025	.879,678	.892,772	.904,745	.914,181	.918,451	.910,632	.881,517
x_1		.046,528	.092,712	.138,364	.183,159	.226,391	.266,149	.289,332
ψ_1		−2,293	−5,040	−8,470	−12,889	−15,672	−28,151	−40,510
x_0	.693,147	.632,910	.567,971	.497,734	.420,667	.334,581	.234,787	.141,779
x_2		−1,933	−4,063	−6,898	−10,547	−15,676	−23,729	−35,047
x_4					3	4	5	16
μ_1		.216,729	.307,351	.377,126	.435,596	.485,733	.527,162	.547,847
μ_3		−677	−2,018	−3,933	−6,477	−9,812	−14,310	−19,174
μ_5		3	24	72	163	317	570	877
μ_7				−2	−6	−15	−27	−60
μ_9						1	2	5
φ_0	.577,350	.560,443	.540,918	.518,075	.490,463	.455,698	.408,046	.351,688
φ_2		−14,450	−29,881	−46,593	−64,857	−85,229	−108,877	−.131,001
φ_4		99	419	1,009	1,930	3,269	5,145	7,040
φ_6		−2	−7	−30	−80	−177	−350	−564
φ_8				1	3	11	30	51
Ω	.707,107	.702,31	.691,96	.673,19	.640,51	.580,55	.451,89	0

These orbits are closed in the plane x, λ but are not necessarily closed in the space (x, λ, φ). To be closed φ_0 must be a commensurable number $\frac{m}{n}$ [13]. In this case when ω increases by $2\pi n$, the orbit closes on itself and the particle will have turned m times around the dipole before closure takes place. Then we shall get a single infinity of periodic orbits for this value of the parameter because all the values that the constant C can take between 0 and $\frac{2\pi}{n}$ give us distinct periodic orbits.

When φ_0 is incommensurable, the orbit will never close on itself and for this value of the parameter we obtain in space one and only one

quasi-periodic orbit which will fill all the surface of revolution having as meridian section the plane periodic orbit (x, λ).

When γ_1 varies continuously, φ_0 varies also continuously and the values of γ_1 for which we have closed periodic orbits will form a set of measure zero.

As we have seen before, it is possible to expand all the coefficients of the periodic orbits in powers of $(a - a^*)^{1/p}$ where a^* is the limiting value

TABLE II (stable)

γ_1	1.000,000	.964,953	.929,898	.895,050	.860,846	.828,245	.800,000
μ	−1.	−.913,378	−.817,609	−.709,708	−.584,751	−.433,313	−.232,786
r_0	.451,305	.470,741	.492,881	.518,615	.549,408	.588,057	.641,268
ω	.611,037	.636,609	.665,459	.698,316	.736,166	.780,440	.833,377
x_1	.177,245	.188,613	.210,458	.232,541	.254,353	.274,700	.290,080
ψ_1	−.100,689	−95,993	−90,684	−84,203	−76,329	−66,885	−54,348
ψ_2	−1,351	−1,078	−737	−417	−364	−189	−63
x_0	6,160	4,894	5,521	9,176	17,918	35,986	74,356
x_2	−.106,874	−.099,519	−.091,545	−.082,810	−.073,091	−.061,986	−.048,576
x_4	−1,703	−1,317	−974	−675	−426	−227	−84
x_6	−48	−33	−22	−13	−7	−4	−1
μ_1	.395,599	.422,446	.448,986	.475,180	.500,743	.524,803	.544,396
μ_3	−24,903	−25,854	−26,515	−26,801	−26,573	−25,565	−23,164
μ_5	755	892	1,019	1,126	1,199	1,212	1,115
μ_7	−51	−61	−72	−81	−89	−91	−82
μ_9	4	5	6	7	8	8	7
φ_0	.145,974	.159,620	.175,866	.195,353	.219,738	.251,355	.296,441
φ_2	−.168,062	−.167,742	−.166,923	−.165,368	−.162,513	−.157,334	−.147,474
φ_4	4,020	5,114	6,153	7,092	7,849	8,310	8,180
φ_6	−453	−518	−596	−664	−721	−756	−716
φ_8	31	40	51	62	70	73	69
φ_{10}	−3	−4	−7	−7	−8	−7	−7

of a. The consideration of periodic orbits very near the limiting one has shown that $p = 2$. We have chosen the positive root for the first branch having the point of equilibrium. The coefficients of the solutions of the variational equations can also be expressed in power series of the same parameter.

Let us write ζ, η for x, λ on the periodic orbit, α and β for the two direction cosines of the periodic orbit; $u\sqrt{\left(\frac{d\zeta}{d\sigma}\right)^2 + \left(\frac{d\eta}{d\sigma}\right)^2}$ for the tangential distance of a neighboring orbit and w for the normal distance; we have:

$$x = \zeta + \frac{d\zeta}{d\sigma} u + \beta w$$

$$\lambda = \eta + \frac{d\eta}{d\sigma} u - \alpha w \qquad (10)$$

The variational equations become

$$\frac{d^2w}{d\sigma^2} + \Theta w = 0 \tag{11}$$

$$\frac{du}{d\sigma} = \frac{2w}{R} \tag{12}$$

where Θ and $\frac{1}{R}$ are functions of the periodic orbit. Let us write

$$\Theta = \theta_0 + 2\theta_2 \cos 2\omega\sigma + 2\theta_4 \cos 4\omega\sigma \cdots \tag{13}$$

Each θ_{2n} is a function of the coefficients of the periodic orbit. If we replace them by their expansion in $(a - a^*)^{\frac{1}{2}}$, we can express θ_{2n} in power series of $(a - a^*)^{\frac{1}{2}}$.

The solution of (11) can be written

$$w = e^{\Omega\sigma}(w_0 + 2w_2 \cos 2\omega\sigma \cdots)$$

We may carry out the calculations of w_0, $w_2 \cdots$ by identification introducing w in the equations and equating to zero the coefficients of the different harmonics $\cos 2n\omega\sigma$. We get then an infinite number of homogeneous equations in w_0, $w_2 \cdots$ and the condition of their compatibility gives the value of Ω. Following Hill's method [14] we have shown [10] that

$$\sin^2 \frac{\pi}{2} \frac{\Omega}{\omega} i = \sin^2 \frac{\pi}{2} \sqrt{\theta_0} \cdot \Delta_1(0) \tag{14}$$

where $\Delta_1(0)$ can be expanded as an integral function of all the coefficients θ_{2n} and Ω^2/ω^2 in power series of $(a - a^*)^{\frac{1}{2}}$. The coefficients w_{2n}, solutions of the homogeneous equations, can also be expressed in the form of an infinite convergent determinant function of the θ_{2n}'s and Ω^2/ω^2 and then can also be expanded in power series of $(a - a^*)^{\frac{1}{2}}$.

By quadrature u will be expressed in the same way and then also x and λ very near the periodic orbit.

If we consider now the general formal solutions for the vicinity of the periodic orbits as were developed by Birkhoff, [15], the different terms of the formal series being essentially solutions of the same linear differential equations (11, 12) with a second member depending on the previous terms of the formal series, then by recurrence they will be expanded in the same way as the first approximation w and u in power series of $(a - a^*)^{\frac{1}{2}}$.

The general solutions near the periodic orbits are of the form

$$x = [x_0 + (x_{0,1}e^{\lambda\sigma} + x_{0,-1}e^{-\lambda\sigma}) + (x_{0,2}e^{2\lambda\sigma} + x_{0,-2}e^{-2\lambda\sigma}) \cdots]$$
$$+ [x_2 + (x_{2,1}e^{\lambda\sigma} + x_{2,-1}e^{-\lambda\sigma}) + (x_{2,2}e^{2\lambda\sigma} + x_{2,-2}e^{-2\lambda\sigma}) \cdots] \cos 2\omega\sigma \cdots$$

and similarly for the other variables, where $\left(\frac{\lambda}{\omega}\right)^2$, x_0, $x_{0,1} \cdots$ are parameters depending also on constants of integration and developable in power series of $(a - a^*)^{\frac{1}{2}}$.

In particular, the asymptotic orbit can be expanded in the same way.

4. Computations and results.

In the computation instead of using $\sqrt{a - a^*}$, it seemed advantageous to use $\mu = \sqrt{\frac{\gamma_1 - \gamma_1^*}{1 - \gamma_1^*}}$ which can be expanded in convergent power series of $(a - a^*)^{\frac{1}{2}}$.

We know from the previous computations [10, 11] the development of the periodic orbits for several values of μ, it will then be possible by interpolation to have all the expansions of the different coefficients.

We write the polar coordinates in the form:

$$\frac{e^2}{2\gamma_1} = r_0 e^{-2z_2\left(\sin^2\omega\sigma + \frac{z_4}{z_2}\sin^2 2\omega\sigma + \frac{z_6}{z_2}\sin^2 3\omega\sigma\right)}$$

$$\sin\lambda = \sqrt{z_1}\sin\left[\omega\sigma + \psi_1\left(\sin 2\omega\sigma + \frac{\psi_2}{\psi_1}\sin 4\omega\sigma\right)\right]$$

$$\varphi - C = \varphi_0\omega\sigma + \varphi_2\sin 2\omega\sigma + \varphi_4\left[\sin 4\omega\sigma + \frac{\varphi_6}{\varphi_4}\left(\sin 6\omega\sigma + \frac{\varphi_8}{\varphi_6}\sin 8\omega\sigma\right)\right]$$

We obtain correct to 5 decimals:

$$r_0 = 0.70550 + \mu\,0.28090 + \mu^2\,0.02000 - \mu^3\,0.00655 + \mu^4\,0.00015$$

$$x_2 = -0.03505 + \mu\,0.05354 - \mu^2\,0.02054 - \mu^3\,0.00010 + \mu^4\,0.00215$$

$$x_4/x_2 = 0.000996\,(1 - \mu)^4 + 0.000700\,(\mu^2 - 1)$$

$$x_6/x_2 = 0.000013\,(1 - \mu)^5$$

$$z_1 = 0.28933 - \mu\,0.04956 - \mu^2\,0.20677 - \mu^3\,0.03262 + \mu^4\,0.00106 - \mu^5\,0.00145$$

$$\psi_1 = -0.04051 + \mu\,0.05731 - \mu^2\,0.01294 - \mu^3\,0.00696 - \mu^4\,0.00310$$

$$\psi_2/\psi_1 = 0.000083\ (1-\mu)^4$$

$$\varphi_0 = 0.35169 + \mu\,0.24079 + \mu^2\,0.00998 - \mu^3\,0.02830 + \mu^4\,0.00320$$

$$-\varphi_2 = 0.13100 - \mu\,0.08403 - \mu^2\,0.04697 + \mu(1-\mu^2)[0.00105 - \mu\,0.00617 - \mu^2\,0.00035 + \mu^3\,0.00116]$$

$$\varphi_4 = 0.00704 - \mu\,0.00201 - \mu^3\,0.00503 - \mu(1-\mu^2)[0.00478 + \mu\,0.00210 - \mu^2\,0.00041 - \mu^3\,0.00051]$$

$$-\varphi_2/\varphi_4 = 0.08000 - \mu\,0.05650 - \mu^2\,0.02350 + \mu(1-\mu^2)[0.01700 + \mu\,0.01800]$$

$$\varphi_2/\varphi_0 = 0.1$$

Let us suppose now that we wish to compute the orbit closing on itself after n oscillations and turning m times around the dipole; then $\frac{m}{n} = \varphi_0$. In the tables I and II giving φ_0 for certain values of μ, φ_0 to correspond to a periodic orbit of the family studied must lie between two φ_0 of the table $\varphi_{0.1} \leq m/n < \varphi_{0.2}$; $\varphi_{0.1}$ corresponding to μ_1 and $\varphi_{0.2}$ to μ_2. By linear interpolation we may find an approximate value of the μ_0. A second approximation will be given by

$$\mu = \mu_0 + \left(\varphi_0 - \frac{m}{n}\right) \frac{1}{\left(\frac{d\varphi_0}{d\mu}\right)_{\mu=\mu_0}}$$

If we continue by iteration, we are able to know the exact value of μ_0 and then all the elements of the periodic orbit.

We have taken as an example $n = 3$, $m = 1$ and we found $\mu = -.076{,}524$, $\gamma_1 = .789{,}779$. Two orthogonal projections of the periodic orbit, one on the equatorial plane and another on one meridian plane, are drawn in Fig. 1.

We have also computed the expansion of $\frac{\Omega}{\omega}$ in series of μ by interpolation between $\frac{\Omega}{\omega}$ computed [10] for the first branch of the periodic orbits $\mu > 0$.

$$\frac{\Omega}{\omega} = \sqrt{\mu}\,[.816{,}50 + (1-\mu)[.305{,}00 - \mu\,.139{,}4 + \mu^2\,.048]]$$

We can then by extrapolation get an approximate value of $\frac{\Omega}{\omega}$ for the other branch. But this value is quite uncertain; we have for $\mu = .232.79$; $\Omega = .596i$; whereas Baños [11] has obtained $.630i$.

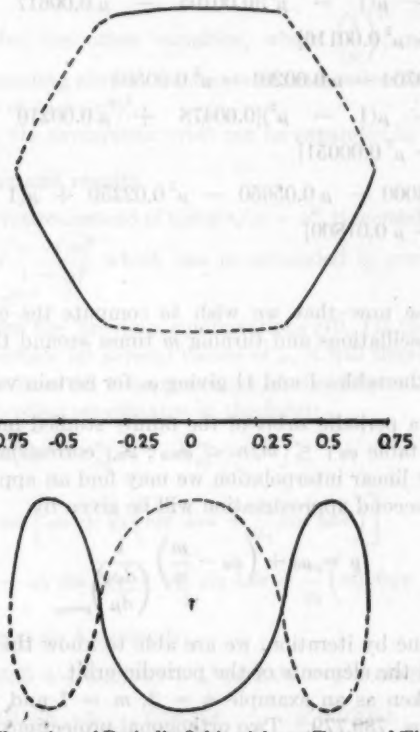

FIG. 1. Plan and Elevation of Periodic Orbit with one Turn and Three Oscillations ($m = 1$, $n = 3$). Scale in Störmer Units

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ON A CLASS OF SINGULAR INTEGRAL EQUATIONS

By Eric Reissner

1. Statement of the Problem. In this note the solution of the integral equation

$$f(x) = h(x) + \int_0^\infty k(x-y)f(y)\,dy \qquad (0 < x < \infty) \tag{1}$$

is given, subject to the restrictions

$$e^{\lambda|x|}k(x) \text{ in } L^2(-\infty, \infty) \text{ for all } \lambda < 1 \tag{2}$$

$$h(x) = 0, \quad x < 0; \qquad h(x) = 0(e^{ax}), \quad 0 < x, \quad a < 1 \tag{3}$$

The homogeneous equation corresponding to (1) and (2) has been solved by Wiener and Hopf[1] and it is shown here that their method may be extended to give the solution of the more general problem.

Since the procedure employed is based on Wiener and Hopf's work, their solution of the homogeneous problem is outlined first. It is then shown that the argument can be generalized to the case of the non-homogeneous equation (1).

2. The Homogeneous Equation. In addition to (2) it is assumed that

$$f(x) = 0(e^{c|x|}),\ 0 < c < 1; \qquad h(x) = 0. \tag{4}$$

Then the modified equation

$$g(x) = f(x) - \int_0^\infty k(x-y)f(y)\,dy \tag{5}$$

is considered where

$$f(x) = 0, \quad x < 0; \qquad g(x) = 0, \quad 0 < x. \tag{6}$$

It follows from (5), (4) and (2) that

$$g(x) = 0(e^{cx}), \quad x < 0. \tag{7}$$

[1] *N. Wiener and E. Hopf*, Ber. Berlin Akad. Wiss. (1931) 696–706. See also *Paley and Wiener*, Fourier Transforms in the Complex Domain, Chap. IV and *Titchmarsh's* Theory of Fourier Integrals, p. 339.

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Writing

$$\left.\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(w)e^{iwz}\,dw \\ g(x) &= \int_{-\infty}^{\infty} G(w)e^{iwz}\,dw \\ k(x) &= \int_{-\infty}^{\infty} K(w)e^{iwz}\,dw \end{aligned}\right\}, \tag{8}$$

we have formally

$$\left.\begin{aligned} F(w) &= \frac{1}{2\pi}\int_{0}^{\infty} f(x)e^{-iwz}\,dx \\ G(w) &= \frac{1}{2\pi}\int_{-\infty}^{0} g(x)e^{-iwz}\,dx \\ K(w) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} k(x)e^{-iwz}\,dx \end{aligned}\right\}. \tag{9}$$

It now follows from (2), (4), (7) that the functions F, G, K are regular and bounded in the following regions of a complex $w = (u + iv)$-plane:

$$\left.\begin{aligned} &F \text{ in } v \leq -c - \epsilon \\ &G \text{ in } -1 + \epsilon \leq v \\ &K \text{ in } -1 + \epsilon \leq v \leq 1 - \epsilon \end{aligned}\right\}. \tag{10}$$

Substituting from the expressions (8) into the integral equation (5), we obtain formally the following equation involving the Fourier transforms (9):

$$G(w) = F(w)[1 - 2\pi K(w)]. \tag{11}$$

Since F, G, K have a common region of regularity given by

$$-1 + \epsilon \leq v \leq -c - \epsilon, \tag{12}$$

the process which led to (11) is justified, and this equation, therefore, subject to the regularity conditions (10), determines F and G. Eq. (11) may be written in the form

$$\log \frac{G(w)}{F(w)} = \log [1 - 2\pi K(w)], \tag{13}$$

where the right side is regular in the strip $S(\epsilon)$: $-1 + \epsilon \leq v \leq -1 - \epsilon$ except at the zeros of $(1 - 2\pi K)$. Denoting these zeros by $w_1, \cdots, w_n$, we obtain a relation between regular functions by writing

$$\log \frac{G(w)}{F(w)} \frac{1}{(w-w_1)\cdots(w-w_n)} = \log \frac{1-2\pi K(w)}{(w-w_1)\cdots(w-w_n)}. \quad (14)$$

In order to make the real parts of the logarithms vanish for $u \to \pm\infty$, a term $\log (w^2+1)^{n/2}$ is added to both sides. To make the imaginary parts vanish also for $u \to \pm\infty$, a term $\log [(w-i)/(w+i)]^m$ is added with m determined such that

$$\log \psi = \log \frac{[1-2\pi K(w)](w^2+1)^{n/2}(w+i)^m}{(w-w_1)\cdots(w-w_n)(w-i)^m} \to 0 \quad \text{as} \quad u \to \pm\infty \quad (15)$$

in the strip $S(\epsilon)$. Next, it can be shown that $|\log \psi|$ belongs to L^2 in $S(\epsilon)$. We may therefore write

$$\log \psi = \frac{1}{2\pi i}\int_{-i\gamma-\infty}^{-i\gamma+\infty} \frac{\log \psi(z)}{z-w}\,dz - \frac{1}{2\pi i}\int_{i\gamma-\infty}^{i\gamma+\infty} \frac{\log \psi(z)}{z-w}\,dz$$

$$= \chi_1(w) - \chi_2(w), \quad \text{say.} \quad (16)$$

It is evident that χ_1 is regular in $-\gamma < v$ and χ_2 in $v < \gamma$. If we write (14) in the form

$$\log G(w)(w+i)^{(n/2)+m} - \log F(w)(w-i)^{m-(n/2)}$$

$$(w-w_1)\cdots(w-w_n) = \chi_1(w) - \chi_2(w) \quad (17)$$

corresponding terms on both sides of the equation have equal domains of regularity. Writing for (17)

$$\frac{G(w)(w+i)^{(n/2)+m}}{\exp(\chi_1)} = \frac{F(w)(w-w_1)\cdots(w-w_n)}{\exp(\chi_2)(w-i)^{(n/2)-m}}, \quad (18)$$

both sides are regular in their respective domains. Since there is a common region of regularity $-1+\epsilon \le v \le -c-\epsilon$ for the functions concerned, it follows that both sides are regular in the entire w-plane and each side must, therefore, be equal to a polynomial. An upper bound for the degree of the polynomial is obtained by taking into consideration the fact that G, F, χ_1 and χ_2 are bounded in their respective regions of regularity, and we see that both sides of (18) can be of degree $\frac{n}{2}+m$ at most. Hence we have

$$\left.\begin{aligned} G(w) &= e^{\chi_1(w)}\frac{P_{(n/2)+m}(w)}{(w+i)^{n/2+m}} \\ F(w) &= e^{\chi_2(w)}\frac{P_{(n/2)+m}(w)(w-i)^{(n/2)-m}}{(w-w_1)\cdots(w-w_n)} \end{aligned}\right\}, \quad (19)$$

where χ_1 and χ_2 are defined by (16) and (15). *Substituting from* (19) *into* (8) *we obtain the solution of the homogeneous problem.*

3. The Non-Homogeneous Equation. It will be sufficient to find one particular integral f_p of (1), since the difference of two solutions must satisfy the homogeneous equation. Instead of (1), we now write

$$g_p(x) = f_p(x) - h(x) - \int_0^\infty k(x-y) f_p(y)\, dy, \tag{20}$$

where g_p and f_p satisfy (6) and (7) and h satisfies (3). In addition to (8) and (9), we assume that

$$h(x) = \int_{-\infty}^{\infty} H(w) e^{iwx}\, dw \tag{21}$$

and because of (3), in $v \leq -a - \epsilon$

$$H(w) = \frac{1}{2\pi} \int_0^\infty h(x) e^{-iwx}\, dx. \tag{22}$$

When (8) and (21) are introduced into (20), we obtain that

$$G_p(w) = F_p(w)[1 - 2\pi K(w)] - H(w). \tag{23}$$

Since H is regular and bounded in $v \leq -a - \epsilon$, F_p in $v \leq -c - \epsilon$ and G in $-1 + \epsilon \leq v$ these three functions have a common domain of regularity given by

$$-1 + \epsilon \leq v \leq -c - \epsilon. \tag{24}$$

To solve Eq. (23) a process is employed which is in part suggested by the method of variation of parameters for non-homogeneous linear differential equations. A suitable assumption is made concerning the polynomial $P_{n/2+m}$ in the solution (19) of the homogeneous equation and then G and F are multiplied by factors to be determined by (23). We write

$$P_{(n/2)+m}(w) = (w + i)^{(n/2)+m} \tag{25}$$

$$\left.\begin{aligned} G_p(w) &= \Gamma(w) e^{\chi_1(w)} \\ F_p(w) &= \phi(w) e^{\chi_2(w)} \frac{(w^2+1)^{n/2}(w+i)^m}{(w-w_1)\cdots(w-w_n)(w-i)^m} \end{aligned}\right\}. \tag{26}$$

Substituting from (26) into (23) it follows from (19) and (11) that

$$\Gamma(w) = \phi(w) - H(w) e^{-\chi_1(w)}. \tag{27}$$

Now $He^{-\chi_1 w}$ is bounded, regular and belongs to L^2 in the strip (24), and we may therefore write

$$H(w)e^{-\chi_1(w)} = \frac{1}{2\pi i}\int_{i\delta'-\infty}^{i\delta'+\infty} \frac{H(z)\exp(-\chi_1(z))}{z-w}\,dz - \frac{1}{2\pi i}\int_{i\delta''-\infty}^{i\delta''+\infty} \frac{H(z)\exp(-\chi_1(z))}{z-w}\,dz$$

$$= \psi_1(w) - \psi_2(w), \quad \text{say} \tag{28}$$

where

$$-1 + \epsilon < \delta' < \delta'' < -c - \epsilon. \tag{29}$$

Plainly ψ_1 is bounded and regular in $\delta' < v$, and ψ_2 in $v < \delta''$. Setting

$$\Gamma(w) = \psi_1(w), \qquad \phi(w) = \psi_2(w), \tag{30}$$

we see that G_p is bounded and regular in $-1 + \epsilon \leq v$ and F_p in $v < -c - \epsilon$, except at the zeros w_ν of $(1 - 2\pi K)$ which lie in the strip $-1 + \epsilon \leq v \leq -c - \epsilon$. *It follows therefore that*

$$\left.\begin{aligned} G_p(w) &= e^{\chi_1(w)}\frac{1}{2\pi i}\int_{i\delta'-\infty}^{i\delta'+\infty}\frac{H(z)\exp(-\chi_1(z))}{z-w}\,dz \\ F_p(w) &= e^{\chi_2(w)}\frac{(w^2+1)^{n/2}(w+i)^m}{(w-w_1)\ldots(w-w_n)(w-i)^m} \cdot \int_{i\delta''-\infty}^{i\delta''+\infty}\frac{H(z)\exp(-\chi_1(z))}{z-w}\,dz \end{aligned}\right\}, \tag{31}$$

where χ_1 *and* χ_2 *are defined by* (16) *and* (18), δ' *and* δ'' *by* (28) and H *by* (22). *Finally*

$$f_p(x) = \int_{-\infty}^{\infty} F_p(w)e^{iwx}\,dw$$

is the required particular solution of the non-homogeneous equation (1).

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THE DISTRIBUTION OF A PRODUCT FROM SEVERAL SOURCES TO NUMEROUS LOCALITIES

By Frank L. Hitchcock

1. Statement of the problem. When several factories supply a product to a number of cities we desire the least costly manner of distribution. Due to freight rates and other matters the cost of a ton of product to a particular city will vary according to which factory supplies it, and will also vary from city to city.

Denote the cost of one ton of product from the i^{th} factory to the j^{th} city by a_{ij} and the number of tons shipped in each case by x_{ij}. The total cost y will then be

$$y = \sum a_{ij}x_{ij} \tag{1}$$

summed for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$ if there are m factories and n cities.

Let the total product shipped from the i^{th} factory to the n cities be f_i; this means we have the equations

$$\begin{aligned} x_{11} + x_{12} + \cdots + x_{1n} &= f_1 \\ x_{21} + x_{22} + \cdots + x_{2n} &= f_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ x_{m1} + x_{m2} + \cdots + x_{mn} &= f_m \end{aligned} \tag{F}$$

which may be called the F-equations for convenience.

Let also the total product consumed by the j^{th} city be c_j tons; this means we have the equations

$$\begin{aligned} x_{11} + x_{21} + \cdots + x_{m1} &= c_1 \\ x_{12} + x_{22} + \cdots + x_{m2} &= c_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ x_{1n} + x_{2n} + \cdots + x_{mn} &= c_n \end{aligned} \tag{C}$$

which may be called the C-equations.

The F-equations and the C-equations taken together form a system of $m + n$ equations which must be satisfied by the mn unknowns x_{ij}. But since the total shipped by factories equals total consumed by cities the sum of the F-equations equals the sum of the C-equations whence

the equations are not independent. It can easily be shown that any *one* equation may be omitted and the rest solved for a properly chosen set of $m + n - 1$ of the x's in terms of the other x's as parameters. If the results are substituted in (1) we shall have an expression for y in terms of $mn - (m + n - 1)$ parameters.

From the nature of the problem none of the unknowns x_{ij} can be negative, and no x_{ij} can exceed the smaller of the two numbers f_i and c_j.

2. Geometrical interpretation. A geometrical picture will be helpful. If the mn unknowns x_{ij} are rectangular coordinates in mn dimensions, then since we have $m + n - 1$ independent equations relating these variables, the region corresponding to possible values of the x's will be of $mn - (m + n - 1)$ dimensions in agreement with the number of parameters found above. Since the possible values of the x's are bounded, on the one hand by zero, on the other by f_i or c_j whichever is smaller, the region of possibilities is a closed region in the space of mn dimensions. This region in which meaningful values of the x's occur will be called for brevity the X-region.

The cost y is, by (1), a linear function of position in the X-region. The least value of y will occur at some point on the *boundary* of the X-region, since a linear function cannot attain its least value in free space.

At any point on the boundary of the X-region at least one of the x's must be zero. For at least one of the x's must have one of its extreme values, otherwise the point would not be on the boundary. And if one of the x's has its maximum value one or more other x's must be zero, by the form of the F-equations and C-equations.

Since the X-region is of dimensions $mn - m - n + 1$, a portion of its boundary over which one of the x's is zero will be of dimensions $mn - m - n$. Taking a point on this part of the boundary of the X-region we may move in a direction of diminishing y until we come to a point where some *other* x reaches a limiting value, and there at least one other x must be zero. The region where two x's are zero will be of dimensions $mn - m - n - 1$. Proceeding in this way we shall eventually reach a point where $mn - m - n + 1$ of the x's are zero, i.e. a fixed point on the boundary of the X-region which may be called a *vertex*.

The value of y at a vertex is not necessarily its least value, but the least value must occur at *some* vertex, which may be called conveniently a *best vertex*. If the particular set of $mn - m - n + 1$ x's which are

zero at a vertex are taken as parameters and the other x's eliminated from (1), the expression for y will take the form

$$y = M + \sum kx \tag{2}$$

where M is a constant and $\sum kx$ denotes a linear combination of the parametric x's. *If none of the coefficients k in this result is negative the vertex is a "best vertex" and M is the required minimum cost.* For $y = M$ at the vertex, and if any parametric x is different from zero the cost y is increased.

The method will be to find a vertex and form the expression for y in terms of the x's which are zero at this vertex. It can be seen by inspection whether we have a "best vertex". If not, we can always pass easily to a better vertex, i.e. one giving a smaller value of y, and continue in this way until we have a best vertex.

3. Finding a vertex. A vertex can always be quickly found in the following manner. Let the F-equations and the C-equations, taken together, be condensed in form according to the scheme

$$\begin{array}{cccccc} x_{11} & x_{12} & \cdots & x_{1n} & f_1 \\ x_{21} & x_{22} & \cdots & x_{2n} & f_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} & f_m \\ c_1 & c_2 & \cdots & c_n & \end{array} \tag{FC}$$

wherein the sum of the x's in a row equals the f on the right and the sum of the x's in a column equals the c at the bottom. We may suppose the factories and the cities to be so numbered that no f is less than any f above it, and no c is less than any c on its left. We now set all the x's of the first row equal to zero except x_{1n} which then equals f_1; *or else* we take all x's of the first column zero except x_{m1} which then equals c_1. *One* of these two procedures is always possible. Suppose the first case, i.e. $x_{1n} = f_1$. We now discard the first row, replacing c_n at the bottom by $c_n - f_1$ assumed to be positive. If this quantity is negative the first procedure is impossible and instead we take $x_{m1} = c_1$, drop the first column, and replace f_m by $f_m - c_1$ which is surely not negative if $c_n - f_1$ is negative as is easily seen.

We have thus reduced the original array to a smaller array, corresponding to one less factory or else one less city, as the case may be. We then continue in just the same way until we have made $mn - m - n + 1$ of the x's zero and so have a vertex.

To illustrate, take an example of four cities and three factories with the array

$$\begin{array}{cccc} x_{11} & x_{12} & x_{13} & x_{14} \quad 25 \\ x_{21} & x_{22} & x_{23} & x_{24} \quad 25 \\ x_{31} & x_{32} & x_{33} & x_{34} \quad 50 \\ 15 & 20 & 30 & 35. \end{array} \tag{3}$$

Either of the procedures mentioned above is possible. Adopting the first we set $x_{11} = x_{12} = x_{13} = 0$, $x_{14} = 25$, drop the first row and replace 35 at the bottom by $35 - 25$ or 10. This gives an array like that for two factories and four cities, namely

$$\begin{array}{cccc} x_{21} & x_{22} & x_{23} & x_{24} \quad 25 \\ x_{31} & x_{32} & x_{33} & x_{34} \quad 50 \\ 15 & 20 & 30 & 10. \end{array}$$

Since the largest number at the bottom is 30, the third column will now play the part of a right hand column but it is hardly necessary to rewrite the array. We merely set $x_{21} = x_{22} = x_{24} = 0$, $x_{23} = 25$. This entails $x_{31} = 15$, $x_{32} = 20$, $x_{33} = 5$, and $x_{34} = 10$. The whole process may be carried out on the original array (3),

$(x_{11})0$	$(x_{12})0$	$(x_{13})0$	$(x_{14})25$	25
$[x_{21}]0$	$[x_{22}]0$	$[x_{23}]25$	$[x_{24}]0$	25
x_{31}	x_{32}	x_{33}	x_{34}	50
15	20	[30]5	(35)10	

where, in the first row, the values to the right of each x are those to be used, the parentheses referring to the first step of the process. In the second row also, the values to the right of each x are to be used, and the brackets refer to the second stage of the work. Finally the values remaining at the bottom are those of the x's of the last row. The whole clearly checks, as is seen by adding rows and columns. The method is perfectly general and may be used in any case to find a vertex.

We now choose the x's which were set equal to zero to be parameters, express the other x's in terms of these, and substitute in the cost expression (1). This is a simple calculation and we find

$$x_{14} = 25 - x_{11} - x_{12} - x_{13} \tag{4}$$

$$x_{23} = 25 - x_{21} - x_{22} - x_{24} \tag{5}$$

$$x_{31} = 15 - x_{11} - x_{21} \tag{6}$$

$$x_{32} = 20 - x_{12} - x_{22} \tag{7}$$

$$x_{33} = 5 - x_{13} + x_{21} + x_{22} + x_{24} \tag{8}$$

$$x_{34} = 10 - x_{24} + x_{11} + x_{12} + x_{13} \tag{9}$$

equations which show by their form that a vertex is obtained when the x's on the right are zero, since the others are left with possible values. The configuration of the X-region and the number of its vertices depend on the constants f and c, not on the costs a_{ij}.

Suppose now the respective costs to be

$$\begin{aligned} a_{11} &= 10, & a_{12} &= 5, & a_{13} &= 6, & a_{14} &= 7, \\ a_{21} &= 8, & a_{22} &= 2, & a_{23} &= 7, & a_{24} &= 6, \\ a_{31} &= 9, & a_{32} &= 3, & a_{33} &= 4, & a_{34} &= 8. \end{aligned} \tag{10}$$

Substituting from (4)–(10) into the cost expression (1) we find

$$y = 645 + 2x_{11} + 3x_{12} + 3x_{13} - 4x_{21} - 4x_{22} - 5x_{24} \tag{11}$$

showing a cost 645 at the vertex where the six parametric x's are zero, not a best vertex since the x's with negative coefficients might be given positive values.

4. Finding a better vertex. We may now let x_{21} increase from zero to its maximum value 15, keeping the other five parameters zero. Inspection of (6) shows then $x_{31} = 0$, i.e. we pass along an edge of the X-region to a new vertex where x_{31} is zero instead of x_{21}. Eliminating x_{21} by aid of (6) we obtain the new system

$$x_{14} = 25 - x_{11} - x_{12} - x_{13} \tag{12}$$

$$x_{23} = 10 - x_{22} - x_{24} + x_{11} + x_{31} \tag{13}$$

$$x_{21} = 15 - x_{11} - x_{31} \tag{14}$$

$$x_{32} = 20 - x_{12} - x_{22} \tag{15}$$

$$x_{33} = 20 - x_{11} - x_{13} - x_{31} + x_{22} + x_{24} \tag{16}$$

$$x_{34} = 10 - x_{24} + x_{11} + x_{12} + x_{13} \tag{17}$$

and

$$y = 585 + 6x_{11} + 3x_{12} + 3x_{13} + 4x_{31} - 4x_{22} - 5x_{24} \tag{18}$$

showing that at the new vertex the cost is 585 or 60 less than at the first vertex.

Continuing in the same way we next eliminate x_{22} and introduce x_{23} as parameter, using (13). At the same time, for brevity, we can eliminate x_{34} and introduce x_{34}, using (17). We obtain

$$x_{14} = 25 - x_{11} - x_{12} - x_{13} \tag{19}$$

$$x_{22} = x_{31} + x_{34} - x_{12} - x_{13} - x_{23} \tag{20}$$

$$x_{21} = 15 - x_{11} - x_{31} \tag{21}$$

$$x_{32} = 20 - x_{31} - x_{34} + x_{13} + x_{23} \tag{22}$$

$$x_{33} = 30 - x_{23} - x_{13} \tag{23}$$

$$x_{24} = 10 - x_{34} + x_{11} + x_{12} + x_{13} \tag{24}$$

and

$$y = 535 + x_{11} + 2x_{12} + 2x_{13} + 4x_{23} + x_{34}\,. \tag{25}$$

The absence of negative signs in the last equation shows that we have a "best vertex" by setting the parametric x's equal to zero, and the required minimum cost is 535. The problem is therefore solved.

The method is quite general and can always be used in such problems. This example, however, presents some interesting features. First we note the absence of the parameter x_{31} from the cost expression (25). It follows that there is some latitude in the manner of distribution. For, keeping the other five parameters zero, we can vary x_{31} from zero to its maximum 15, with consequent changes in x_{23}, x_{21}, and x_{32}, seen from (20), (21), and (22), without altering the cost from 535. The cost is constant all along an edge of the X-region.

Moreover setting $x_{31} = 15$, still keeping the other five parameters zero, will entail $x_{21} = 0$, by (21). We have arrived at an extremity of the edge along which y is constant, and have another "best vertex" with, of course, the same cost. That is to say, a "best vertex" need not be unique.

Another interesting feature of this example is that, setting the six parametric x's of (19)–(25) equal to zero we *also* have $x_{22} = 0$, that is we have *seven* of the x's zero at this best vertex. Since *any six* of these could be used as parameters, we have various ways for expressing y at this vertex. Suppose we decide to use x_{22} as parameter instead of x_{34}, which we can eliminate by aid of (20). We get

$$y = 535 + x_{11} + 3x_{12} + 3x_{13} + 5x_{23} + x_{22} - x_{31}\,. \tag{26}$$

We might at first think that, by giving to x_{31} a positive value, we could obtain a cost less than 535, contradicting previous results. But if the other five parameters in (26) are kept zero we cannot increase x_{31} because by (20), this would entail a negative value for x_{34}. In other words the *absence* of negative signs in the expression for y at a vertex is sure sign of a "best vertex" but the *presence* of a negative is not sure proof of the contrary. The test, as just illustrated, is whether we are able to *increase* a parameter having a negative coefficient.

A METHOD OF SUCCESSIVE APPROXIMATIONS OF EVALUATING THE REAL AND COMPLEX ROOTS OF CUBIC AND HIGHER-ORDER EQUATIONS

By Shih-Nge Lin

Introduction

In the study of the differential equations determining the stability of the disturbed motions due to gusts in aeronautics, it is required to solve the real and complex roots of algebraic equations of quartic and higher order (1), (2), (3). Zimmerman (4) suggested a semigraphical method which can solve the quartic equation without knowing the nature of the roots. The method, though reasonably convenient for solving quartic equations, is not applicable for higher order ones.

Lyon (5), and Ku (6) have presented practical methods for evaluating the complex roots of quartic equations, and Woodruff (7) extended the method to solve sextic and higher order equations. There are also the well known Graeffe's (8) root squaring method and the Horner-G. C. D. method by Hitchcock (9).

The following method is applicable, in many cases for evaluating both real and complex roots of algebraic equations of any order. The process involved in reaching solutions within engineering accuracy by using an ordinary slide rule is, when applicable, usually less laborious than the previous methods known.

Principle

For an algebraic equation of the nth order,

$$x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n = 0 \qquad (1)$$

if n is odd, there is at least one real root which can usually be solved by Horner-Newton's method without much difficulty. Thereby, the equation can be reduced to one of even order. If an equation of even order can be factored into quadratic factors, the roots, either real or complex, can be easily obtained.

The process of successive approximations is composed of repeated divisions designed to separate the given equation into quadratic factors.

If one pair of roots, real or complex, is small in absolute magnitude compared with the other pairs, then,

$$x^2 + cx + d = 0 \tag{2}$$

with

$$c = a_{n-1}/a_{n-2}$$

$$d = a_n/a_{n-2}$$

can be taken as a first approximation for the factorization of (1) by the process of division. Assume that (1) is factored, in general, into,

$$(x^{n-2} + b_2x^{n-3} + \cdots + b_{n-1}x + b_n)(x^2 + cx + d) + Px + Q = 0 \tag{3}$$

with $Px + Q$ as the remainder. If the remainder of the above division is zero, the factorization of (1) is then successful. However, the remainder is, in general, not even nearly zero, unless c and d are extremely small compared with b_n, b_{n-1} and so on. However, by repeating the factorization of (1) using the factor

$$(x^2 + c_1x + d_1)$$

where

$$c_1 = (a_{n-1} - d \cdot b_{n-1})/b_n$$

$$d_1 = a_n/b_n$$

instead of

$$(x^2 + cx + d),$$

it is found that the remainder is nearer to zero than before. The process is then said to be "convergent" and can be repeated as many times as is necessary to make the remainder nearly zero within allowable percentage of error.

If the process is divergent, that is, the remainder is further away from being zero, the given equation should then, be transformed either by Graeffe's root squaring method or by Horner's root diminishing process so that the new equation can then be factored by the above process with convergency.

Illustrative Examples

In order to show the superiority of this method over the previous processes known and to illustrate the correct steps in obtaining the solutions, the following examples are worked out.

EXAMPLE 1. Take the quartic equation,

$$x^4 + 10.65x^3 + 89x^2 + 15.5x + 27 = 0.$$

For first approximation, assume the factor,

$$x^2 + cx + d$$

with

$$c = 15.5/89 = 0.174$$

$$d = 27/89 = 0.304$$

as the trial divisor for the factorization of the given equation. The process of division is then carried out in the usual manner as follows,

divisor	given equation
$1 + .174 + .304$	$1 + 10.65 + 89.0 + 15.5 + 27.0$
	$1 + .174 + .304$
	$10.476 + 88.696 + 15.5$
	$10.476 + 1.8 + 3.18$
	$86.9 + 12.32 + 27.0$
	$86.9 + 15.10 + 26.4$
	$-2.78 + 0.6$

Thus, the given equation is factored into,

$$(x^2 + 10.476x + 86.9)(x^2 + .174x + .304) - 2.78x + .60 = 0$$

with

$$b_{n-1} = 10.476$$

$$b_n = 86.9 \quad \text{and } d \cdot b_{n-1} = 3.18 \text{ and } a_{n-1} - d \cdot b_{n-1} = 12.32.$$

For second approximation, using the factor,

$$x^2 + c_1x + d_1$$

where

$$c_1 = 12.32/86.9 = .142$$

$$d_1 = 27/86.9 = .311$$

as the trial divisor. The reason is obvious as we see that if b_n remains to be 86.9, the remainder would then be zero. Repeating the division in a briefer form as follows,

$$
\begin{array}{rl|l}
 & \text{divisor} & \text{given equation} \\
1 + .142 + .311 & | & 1 + 10.65 + 89.0 + 15.5 + 27.0 \\
 & & \quad .142 + .311 \\
\hline
 & & 10.51 + 88.69 + 15.5 \\
 & & \quad 1.49 + 3.27 \\
\hline
 & & 87.2 + 12.23 + 27.0 \\
 & & \quad 12.36 + 27.1 \\
\hline
 & & -.13 - 0.10
\end{array}
$$

It is seen that the remainder is $-.13x - 0.10$ which is very nearly equal to zero, (within one per cent of $15.5x + 27$). The given equation can then be considered to be factored into,

$$(x^2 + 10.51x + 87.2)(x^2 + .142x + .311) = 0$$

within one per cent of error. For still closer approximation, the process can be repeated by using the divisor,

$$x^2 + c_2x + d_2$$

with

$$c_2 = 12.23/87.2 = .1405$$

$$d_2 = 27/87.2 = .310$$

$$
\begin{array}{rl|l}
1 + .1405 + .310 & | & 1 + 10.65 + 89.0 + 15.5 + 27.0 \\
 & & 1 + .1405 + .310 \\
\hline
 & & 10.5095 + 88.69 + 15.5 \\
 & & \quad 1.48 + 3.26 \\
\hline
 & & 87.21 + 12.24 + 27.0 \\
 & & \quad 12.3 + 27.0 \\
\hline
 & & -.06 + 0.0
\end{array}
$$

Therefore, the remainder is even closer to zero than before. And the solution which gives two pairs of complex roots within probably .5% of error can be obtained from the factored quadratics,

$$(x^2 + 10.5095x + 87.21)(x^2 + .1405x + .310) = 0$$

as $-5.255 \pm 7.72i$ and $-0.07025 \pm .552i$.

It is seen that as the remainder is nearer to zero in the successive divisions, the trial divisors become closer to each other. Therefore, without checking the remainder in each step, the convergency can be obtained by examining the successive trial divisors.

EXAMPLE 2. Take the equation,

$$x^4 + 10.65x^3 + 129x^2 + 203.5x + 70 = 0.$$

Following exactly the same process as illustrated in Example 1,

No. of approximations	trial divisors $(x^2 + cx + d)$ c	d
first	1.58	.543
second	1.74	.613
third	1.755	.620
fourth	1.755	.620

Therefore, the given quartic is nearly exactly factored into,

$$(x^2 + 8.91x + 112.9)(x^2 + 1.755x + .520) = 0$$

which gives one pair of complex roots, $-4.455 \pm 9.65i$ and one pair of real roots, -1.3775, and $-.3775$.

This shows that the method can be used to find real roots as well as complex ones.

EXAMPLE 3. Take Woodruff's example (reference 7),

$$x^6 - 22x^5 + 10{,}521x^4 - 140{,}628x^3 + 3{,}920{,}868x^2 + 27{,}391{,}840x + 65{,}104{,}000 = 0.$$

As an ordinary slide rule is to be used, it is advisable to keep only four significant figures of the coefficients of the given equation. And to save space, the roots of the given equation are divided by 10, giving

$$x^6 - 2.2x^5 + 105.2x^4 - 140.6x^3 + 392.1x^2 + 273.9x + 65.1 = 0.$$

Following the process as described before, we get,

Approximations	c	d
first	.698	.166
second	.590	.124
third	.599	.130

The fourth and final approximation factors the equation into, $(x^4 - 2.8x^3 + 107x^2 - 204.4x + 500.9)(x^2 + .60x + .13) = 0$. The quartic factor is then again factored, as before (after three approxima-

tions) giving the final solution as $(x^2 - 2x + 4.99)(x^2 - .80x + 100.4)$ $\cdot(x^2 + .60x + .13) = 0$. Multiplying back the roots by ten, we have

$$(x^2 - 20x + 499)(x^2 - 8x + 10040)(x^2 + 6x + 13) = 0.$$

Comparing with the exact solution,

$$(x^2 - 20x + 500)(x^2 - 8x + 10016)(x^2 + 6x + 13) = 0$$

it is noted that neglecting the less important significant figures in the given equation introduces only very slight difference in the final solution.

EXAMPLE 4. Take the equation in Lyon's example (reference 5)

$$x^4 - 6x^3 + 47x^2 - 18x + 290 = 0.$$

Following the previous process, we find,

Approximations	c	d
1st	−.383	6.17
2nd	.422	7.50
3rd	.713	6.86
4th	.626	6.46
5th	.542	6.34
6th	.530	6.56
7th	.565	6.60
8th	.573	6.58
9th	.570	6.56
10th	.567	6.56
11th	.568	6.56
12th	.566	6.56
13th and final	.566	6.56

The equation is then factored into,

$$(x^2 - 6.566x + 44.15)(x^2 + .566x + 6.56) = 0.$$

This example illustrates the case of a slowly convergent process. However, if the given equation is transformed by Graeffe's root squaring process *only once*, giving the new equation as,

$$x^4 - 58x^3 + 2573x^2 - 26936x + 84100 = 0.$$

The new equation can then be factored as before,

Approximations	c	d
first	−10.45	32.70
second	−12.50	41.10
third	−12.80	42.80
fourth and final	−12.81	43.10

giving the solution as

$$(x^2 - 45.19x + 1950.9)(x^2 - 12.81x + 43.10) = 0.$$

Using the reverse process of root-squaring for each one of the quadratic factors, we obtain the solution for the given equation as, $(x^2 + ax + b)(x^2 + cx + d) = 0$ where a, b, c, d can be determined as follows,

$$a^2 - 2b = -45.19 \quad \text{and} \quad b^2 = 1950.9$$

$$c^2 - 2d = -12.81 \quad \text{and} \quad d^2 = 43.10.$$

Thus, $b = \sqrt{1950.9} = +44.15$ $d = \sqrt{43.10} = +6.56$

$$a^2 = 2b - 45.19, \quad \text{giving} \quad a = -6.56,$$

$$c^2 = 2d - 12.81, \quad \text{giving} \quad c = +.557.$$

b and d must be positive in order that a and c be real. The sign of a and c is determined by noting that $a + c$ must be equal to -6.0, the coefficient of x^5 of the given equation. Thus the root-squaring process improves the convergency of the successive approximations.

EXAMPLE 5. Take Hitchcock's example (reference 9),

$$x^8 - 3.012x^7 + 3.225x^6 + 1.021x^5 + 6.986x^4 - 21.887x^3 + 8.110x^2 + 5.901x + 23.889 = 0.$$

Assume that the given equation can be factored into

$$(x^4 + Bx^3 + Cx^2 + Dx + E)(x^4 + bx^3 + cx^2 + dx + e) = 0.$$

If the coefficients b, c, d and e are comparatively smaller than B, C, D and E, we divide the given equation by the trial divisor $x^4 + bx^3 + cx^2 + dx + e$ with,

$$b = a_{n-3}/a_{n-4} = -21.89/6.99 = -3.13$$

$$c = a_{n-2}/a_{n-4} = 8.11/6.99 = 1.162$$

$$d = a_{n-1}/a_{n-4} = 5.90/6.99 = .845$$

$$e = a_n/a_{n-4} = 23.89/6.99 = 3.42$$

we have,

$$1 - 3.13 + 1.162 + .845 + 3.42$$

as a trial divisor. Performing the division as follows,

$$
\begin{array}{l}
1 - 3.012 + 3.225 + 1.012 + 6.990 - 21.89 + 8.11 + 5.90 + 23.89 \\
1 - 3.127 + 1.162 + 0.845 + 3.42 \\
\hline
\quad .115 + 2.063 + 0.176 + 3.57 - 21.89 \\
\quad -.360 + .134 + .97 + .39 \\
\hline
\quad 2.423 + .042 + 2.60 - 22.28 + 8.11 \\
\quad -7.60 + 2.82 + 2.05 + 8.29 \\
\hline
\quad 7.642 - .22 - 24.33 - .18 + 5.90 \\
\quad -23.90 + 8.89 + 6.45 + 26.1 \\
\hline
\quad 23.68 - 33.22 - 6.63 + 20.2 + 23.89
\end{array}
$$

For second approximation we use,

$$
\begin{aligned}
b &= -33.22/23.68 = -1.4, \\
c &= -6.63/23.68 = -.28 \\
d &= -20.2/23.68 = -.85 \\
e &= 23.89/23.68 = 1.01
\end{aligned}
$$

as trial divisor, repeating the same process, we obtain the following results,

Approximations	b	c	d	e
1st	−3.13	1.162	.845	3.42
2nd	−1.14	−.28	−.85	1.01
3rd	−1.95	1.01	.287	2.54
4th	−2.45	.82	.07	2.75
5th	−1.81	.385	−.375	1.83
6th	−2.06	.805	.03	2.39
7th	−2.13	.67	−.107	2.32
8th	−2.02	.634	−.132	2.20
9th	−2.05	.685	−.089	2.26
10th	−2.11	.69	−.073	2.31
11th	−2.06	.65	−.108	2.24
12th	−2.07	.675	−.081	2.28
13th	−2.09	.675	−.087	2.30
14th	−2.07	.665	−.095	2.27
15th	−2.08	.670	−.087	2.29
16th obtained by using average of 14th and 15th	−2.075	.670	−.090	2.28
final	−2.075	.668	−.091	2.28

Therefore, the given equation is factored into

$$(x^4 - .937x^3 + .613x^2 + 3.01x + 10.466)$$
$$(x^4 - 2.075x^3 + .668x^2 - .091x + 2.28) = 0.$$

The two quartics can then be easily solved either by the method of root squaring and then using successive approximations as illustrated in Example 4, or by the following root diminishing transformations.

(A) To factor the quartic $x^4 - 2.075 + .668x^2 - .091x + 2.28 = 0$.

The factorization by successive approximations without transforming the equation is unsuccessful as the process is divergent. However, if the equation is transformed so that its roots are diminished by 2 by Horner's method, we get the new equation

$$x^4 + 5.925x^3 + 12.22x^2 + 9.68x + 4.17 = 0$$

the process of successive approximations is then convergent,

Approximations	c	d
1st	.79	.34
2nd	1.01	.53
3rd	1.06	.62
4th	1.036	.65
5th	.97	.63
6th and final	.97	.615

The transformed equation is then factored into

$$(x^2 + .97x + .615)(x^2 + 4.96x + 6.795) = 0.$$

Increasing back the roots of each factor by two we get

$$(x^2 - 3.03x + 2.675)(x^2 + .96x + .875) = 0$$

giving two pairs of complex roots, $1.515 \pm .618i$ and $-.48 \pm .804i$ as the solution.

(B) The remaining quartic $x^4 - .937x^3 + .613x^2 + 3.01x + 10.47 = 0$ can be solved in a similar way by first being transformed into a new equation by increasing the roots by 2, giving,

$$x^4 - 8.94x^3 + 30.24x^2 - 42.69x + 30.39 = 0$$

which can be factored into,

$$(x^2 - 1.92x + 2.08)(x^2 - 7.02x + 14.68) = 0.$$

Diminishing the roots back by 2, we have

$$(x^2 + 2.08x + 2.24)(x^2 - 3.02x + 4.68) = 0$$

giving another two pairs of complex roots, $-1.04 \pm 1.08i$ and $1.51 \pm 1.55i$ as the solution. It is interesting to note that one pair of the roots found by Hitchcock is,

$$-1.042869 \pm 1.070971i$$

as compared with

$$-1.04 \pm 1.08i,$$

the error is less than one per cent despite so many transformations made.

It is interesting to note that the given equation can also be solved by factoring into a sextic and a quadratic factor as follows. After squaring the roots of the given equation, we get

$$x^8 + 2.61x^7 + 30.52x^6 + 71.62x^5 + 229.2x^4 + 224.8x^3 + 668.2x^2 - 352x + 573 = 0.$$

After nine approximations, the equation can be factored into,

$$(x^6 + 3.43x^5 + 32.6x^4 + 95.8x^3 + 283.7x^2 + 386.6 + 775.6)(x^2 - .82x + .738) = 0.$$

The quadratic factor, after the reverse process of root-squaring gives $x^2 + .95x + .86 = 0$ which is the pair of roots having the smallest absolute magnitude.

Discussions and Conclusions

From the previous examples solved, it is seen that the method of successive approximations can be used to solve various types of algebraic equations of higher orders. For cases where the process is moderately convergent, this method is more rapid than previous methods known. For cases, where suitable transformations are necessary, the speed in arriving at the final solution depends somewhat on the experience of the investigator. Even for an extremely difficult case, such as that of Example 5, an experienced investigator could still solve the equation faster than by previous known methods.

The skill in transforming the given equation into a new one which is convergent for the process, is not so hard to obtain as it might seem to be; for the convergency of the process depends largely upon the single

fact that b_n in (3) for successive divisions should not vary too much. With this fact in mind, the given equation can then be transformed to suit the purpose. In general, transformation by Horner's root diminishing process is easier to perform than Graeffe's root squaring process but requires more experience to make correct judgment. In any case, the convergency of the process can be obtained by squaring the roots of the given equation three or four times, but long before the roots are "separated" as required by Graeffe's method.

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Editor's note. In a letter from Chengtu, China, Dr. Lin suggests an improvement which is in substance as follows. Let a trial divisor be $x^2 + d_1x + d_0$ and write

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$
$$= (x^{n-2} + q_{n-3}x^{n-3} + \cdots + q_1x + q_0)(x^2 + d_1x + d_0) + r_1x + r_0. \quad (1)$$

In the above paper we would take for a new divisor

$$x^2 + \frac{a_1 - q_1d_0}{q_0}x + \frac{a_0}{q_0}; \quad (2)$$

if, however, we use, instead of d_0 in the numerator of the coefficient of x, the new value $\frac{a_0}{q_0}$, the convergence is much improved. Thus the new divisor is

$$x^2 + \frac{a_1 - \frac{a_0q_1}{q_0}}{q_0}x + \frac{a_0}{q_0} \quad (3)$$

instead of (2). A similar improvement can be made when we use a trial divisor of higher degree.

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A COMPARISON OF METHODS FOR EVALUATING THE COMPLEX ROOTS OF QUARTIC EQUATIONS

By Henry S. Sharp

The study of damped vibrations in electrical and mechanical systems leads to linear differential equations of high even order. The associated algebraic equations have only complex roots. This paper is an effort to collect in one place all the practical methods for the solution of fourth degree equations of this kind. Such a survey will permit a comparison of relative merit and facilitate the choice of the most efficient method for a particular problem.

The explicit solution of Ludovici Ferrari given in most texts on the theory of equations will not be considered here; for the complete accuracy, which is its only advantage, is usually not required, or, in any case, is scarcely worth the great labor involved.

Notation. The general quartic equation with real coefficients will be indicated by:

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0 \tag{1}$$

Its roots will be represented as convenient by either or both of the equivalent forms:

$$x_1 = u_1 + iv_1 = r_1(\cos\phi_1 + i\sin\phi_1)$$
$$x_2 = u_1 - iv_1 = r_1(\cos\phi_1 - i\sin\phi_1)$$
$$x_3 = u_2 + iv_2 = r_2(\cos\phi_2 + i\sin\phi_2)$$
$$x_4 = u_2 - iv_2 = r_2(\cos\phi_2 - i\sin\phi_2)$$

where $i = \sqrt{-1}$

The quadratic factors of (1) will be represented by:

$$(x^2 + a_1x + b_1)(x^2 + a_2x + b_2) = 0 \tag{2}$$

So that:

$$a_1 = -2u_1, \qquad a_2 = -2u_2, \qquad b_1 = u_1^2 + v_1^2, \qquad b_2 = u_2^2 + v_2^2$$

The relations between the coefficients and the roots will be numbered for reference as follows:

$$A = -2(u_1 + u_2) \tag{3}$$

$$\begin{aligned} B &= +(u_1^2 + v_1^2) + 4u_1u_2 + (u_2^2 + v_2^2) \\ &= +r_1^2 + 4u_1u_2 + r_2^2 \end{aligned} \tag{4}$$

$$\begin{aligned} C &= -2u_1(u_2^2 + v_2^2) - 2u_2(u_1^2 + v_1^2) \\ &= -2(u_1r_2^2 + u_2r_1^2) \end{aligned} \tag{5}$$

$$\begin{aligned} D &= +(u_1^2 + v_1^2)(u_2^2 + v_2^2) \\ &= +r_1^2r_2^2 \end{aligned} \tag{6}$$

Types of Solutions. The methods of evaluating the complex roots of equation (1) may be broadly classified into four types, defined as follows:

Type I. Methods devised to evaluate the absolute values (r_1, r_2) and the amplitudes (ϕ_1, ϕ_2) directly.

Type II. Methods devised to find the coefficients a_1, a_2, b_1, and b_2 so that the complex roots may be found by the quadratic formula.

Type III. A method of determining u_1, u_2, v_1, and v_2 directly.

Type IV. Methods of approximate solution depending on the relative magnitude or other special characteristics of the roots, followed by an improvement or correction of the approximation.

Type I

The Method of Graeffe. For an exposition of the theory behind Graeffe's method see "Vorlesungen Uber Numerisches Rechnen," by C. Runge and H. Kŏnig. In brief, the method as applied to the quartic equation is this: if the equation:

$$z^4 + A'z^3 + B'z^2 + C'z + D' = 0 \tag{7}$$

is derived from the coefficients of equation (1) by the formulae

$$\begin{aligned} A' &= A^2 - 2B \\ B' &= B^2 - 2AC + 2D \\ C' &= C^2 - 2BD \\ D' &= D^2 \\ z &= x^2 \end{aligned}$$

then the roots of (7) are the squares of the roots of (1). If the roots of (1) differ in absolute value, the roots of equation (7) must differ by a greater amount. (The case of roots with equal absolute values is easily detected and solution is possible by a modification of Graeffe's method or, more easily, in another way to be considered later.)

If the transformation of equation (1) into equation (7) is repeated on equation (7), an equation will result with roots that are the fourth power of the roots of (1). Continuing in this way, an equation may be found with roots that differ in absolute value as greatly as may be desired. It can be seen from equation (4) that when the absolute value of one root of an equation is very much greater than the absolute value of the other, the coefficient of x^2 will be nearly the square of the absolute value of the larger root. When, by Graeffe's process, an equation has been obtained with roots that are the $2n$th power of the roots of the given equation, the coefficient of x^2 is nearly the $4n$th power of the absolute value of the larger root of the given equation. The greater the number n the more nearly this is true.

The number n, i.e., the number of times the root squaring process is performed, depends on the particular equation and the accuracy desired. The root squaring is terminated when the coefficient of x^2 in the last derived equation is equal to the square of the coefficient of x^2 in the previous step to the number of significant figures desired.

Having found in this manner the $4n$th power of the larger absolute value the square of the absolute value can be found most easily by the use of logarithms. Then equation (6) gives the square of the absolute value of the smaller root. Having these values, equations (3) and (5) become a pair of linear equations for the determination of u_1 and u_2. Then the relation $v^2 = r^2 - u^2$ determines the imaginary parts.

While Graeffe's method is simple in theory it is subject to certain practical limitations. There is no accurate way of determining beforehand how much the roots differ in absolute value, and, consequently, no way of telling how many times the root squaring process will have to be repeated. There is a loss in accuracy of about one significant figure for each two repetitions of the root squaring. Hence the process must be started with several more significant figures than are needed in the final result. Just how many more can not be accurately determined. Another disadvantage of the method is that an error in the early steps is increased many times and there is no means of checking for such errors until the final calculations have been made. Since the root squaring process is a matter of substitution and resubstitution with

changes in algebraic signs and a rather tricky shifting of decimal points, there is every opportunity for numerical error.

The Method of W. V. Lyon. Another method of Type I is due to W. V. Lyon and is published in the "Journal of Mathematics and Physics," April 1924, No. 3, Vol. III. The derivation of this method depends on the treatment of a complex root as a vector in a real plane. In this sense the quartic equation is a statement that a certain vector sum is zero. By separately equating to zero the sums of the components parallel and perpendicular to the middle vector, two equations are obtained which must be satisfied by r and ϕ. For the quartic equation the direction of the middle vector is the direction of the vector: $r^2/2\phi$.

In substance the method as applied to equation (1) is this: if m is the positive real root of the cubic equation

$$m^3 - Bm^2 + (AC - 4D)m - (A^2D - 4BD + C^2) = 0$$

the roots of equation (1) can be found from

$$r^2 = \frac{m \pm \sqrt{m^2 - 4D}}{2} \tag{8}$$

$$\cos \phi = \frac{r(Ar^2 - C)}{2(D - r^4)} \tag{9}$$

Equation (8) gives two values r_1 and r_2 with which ϕ_1 and ϕ_2 can be found from (9) and the real and imaginary parts may then be computed.

It is evident from equations (6) and (8) that the quantity m is

$$m = r_1^2 + r_2^2$$

and that when the two absolute values are somewhat different in magnitude, equation (8) requires the subtraction of two nearly equal numbers. This means that one value of r will be obtained with more accuracy than the other. A similar situation will frequently arise in the numerator of the right side of equation (9). For that reason m must be found from the cubic with two or three more significant figures than are desired in the final result.

The method of W. V. Lyon has several advantages over the method of Graeffe, especially for the quartic equation. The advantage lies principally in the simplicity of the coefficients of the resolved cubic and in the fact that the real root of this equation can be found by Horner's method. Any method wherein a considerable part of the calculation

can be performed by synthetic division is to be preferred over a method which does not lend itself to this very efficient process.

The method has the disadvantage inherent in Type I, that considerable computation is required after the determination of r_1 and r_2. In practice it is the numerical values of the real and imaginary parts that is desired.

The foregoing are the only methods of Type I that this writer has been able to find. There are several related methods which determine the absolute values and amplitudes, but these rightly come under Type IV because of the nature of the process employed.

Type II

The Method of Y. H. Ku. In the "Journal of Mathematics and Physics," February 1926, No. 2, Vol. V, an article of Y. H. Ku describes a method of determining the coefficients of the quadratic factors of equation (2). This method depends upon the same resolved cubic as does the method of W. V. Lyon.

Since $m = r_1^2 + r_2^2$, it is evident that the two values of r^2 obtained from equation (8) are the coefficients b_1 and b_2 of equation (2). Mr. Ku then finds a_1 and a_2 from two equations that are most easily derived from equations (3) and (4). They are

$$a_1 = \frac{A - \sqrt{A^2 - 4B + 4m}}{2}$$

$$a_2 = \frac{A + \sqrt{A^2 - 4B + 4m}}{2}$$

The roots are then found by applying the quadratic formula to the quadratic factors.

The two methods are identical for practical purposes. If there is an advantage one way or the other, it lies with the method of W. V. Lyon; for the labor of computing ϕ_1 and ϕ_2 is no more than that of computing a_1 and a_2, and the subsequent calculation of the real and imaginary parts of the roots is certainly easier than using the quadratic formula on the large numbers involved. Mr. Ku, in the article cited, shows that his method is applicable to quartic equations with complex coefficients and extends the method to sixth order equations.

There are two other methods of resolving a quartic equation into quadratic factors, one due to Ferrari and the other to Descartes. (See Chapt. VI, "Theory of Equations," by W. S. Burnside and A. W.

Panton.) These methods also depend on the solution of a resolved cubic, but a cubic with exceedingly complicated coefficients, so that neither method is at all satisfactory for practical use.

A New Method. A method of Type II was discovered by this writer in the course of another investigation. It is of little practical importance and is included here mostly as a matter of record.

If equation (1) has the quadratic factor:

$$x^2 + ax + b$$

the remainder after division must be identically zero. This leads to two equations which must be satisfied by the coefficients a and b, namely:

$$b^2 - (a^2 - Aa + B)b + D = 0 \tag{10}$$

$$b = \frac{a^3 - Aa^2 + Ba - C}{2a - A} \tag{11}$$

If b is eliminated between these two equations, there results:

$$a^6 - 3Aa^5 + (3A^2 + 2B)a^4 - A(A^2 + 4B)a^3 + (2A^2B + AC + B^2 - 4D)a^2 - A(AC + B^2 - 4D)a - (A^2D - ABC + C^2) = 0 \tag{12}$$

This equation is too much for easy application; but if the given equation is first reduced to the form:

$$z^4 + B'z^2 + C'z + D' = 0 \tag{13}$$

by the substitution:

$$x = z - \frac{A}{4}$$

which can be most easily accomplished by using Horner's method to reduce the roots of equation (1) by $-A/4$; and if we write:

$$p = a^2$$

then equation (12) becomes:

$$p^3 + 2B'p^2 + (B'^2 - 4D')p - C'^2 = 0 \tag{14}$$

The positive real root of this cubic gives two values of a. Then the equation

$$b = (a^3 + B'a - C')/2a \tag{15}$$

will furnish the corresponding values of b. We then have the two quadratic factors of equation (13). Solving these by the quadratic formula, the roots of (1) are found from

$$x = z - \frac{A}{4}$$

The main utility of this method is that it illustrates two principles, (1) that the remainder when (1) is divided by one of its quadratic factors must be identically zero, and (2) that (13) may be derived from (1) by Horner's method of root reduction; both of which will be used later.

Type III

The single method of Type III is the direct or, perhaps, the natural method; for it determines the real and imaginary parts of the complex roots directly. It was developed by this writer as an adaptation of Sylvester's "Dialytic Method of Elimination" particularly suited to the solution of quartic equations with only complex roots.

If equation (1) has a complex root: $x = u + iv$ then substitution of this value in (1) followed by equating real and imaginary parts to zero, produces two equations which must be satisfied by u and v, namely:

$$v^4 - (6u^2 + 3Au + B)v^2 + (u^4 + Au^3 + Bu^2 + Cu + D) = 0 \quad (16)$$

$$(4u + A)v^2 - (4u^3 + 3Au^2 + 2Bu + C) = 0 \quad (17)$$

If v is eliminated between (16) and (17) a sixth degree equation results. But this sixth degree equation will become a cubic in u^2 if we use the procedure of the last Type II example.

That is, we transform equation (1) to equation (13) by the substitution: $x = z - A/4$; using Horner's process for reducing the roots of (1) by $-A/4$. It is then apparent that equations (16) and (17) become:

$$v^4 - (6u^2 + B')v^2 + (u^4 + B'u^2 + C'u + D') = 0 \quad (18)$$

$$v^2 = (4u^3 + 2B'u + C')/4u \quad (19)$$

And now v is eliminated from (18) and (19) to obtain the cubic

$$64p^3 + 32B'p^2 + 4(B'^2 - 4D')p - C'^2 = 0 \quad (20)$$

where:

$$p = u^2$$

The procedure is, in brief:

1) Transform the given equation (1) to the form (13), thus obtaining the coefficients: B', C', and D'.
2) Compute the coefficients of equation (20).
3) Solve (20) for p, using Horner's method, to the desired degree of accuracy.
4) Extract the square root of p to obtain two values of u and use these in (19) to obtain the corresponding values of v. Synthetic division aids in the substitution of u in (19).
5) The two values of $u + iv$ which may now be formed are the roots of (13). Add $-A/4$ to them to obtain the two roots of equation (1). The conjugate roots are found in the same way from $u - iv$.

This method may, at first glance, seem somewhat complicated, but steps 1, 3, and 4 may all be done by synthetic division and steps 2 and 5 are not particularly difficult unless very large numbers are involved. Because synthetic division can be used for most of the calculations the method just described is the least laborious so far considered.

To test the possibilities of the method the roots of the equation

$$x^4 + 8.490x^3 + 24.50x^2 + 3.385x + 0.9170 = 0 \qquad (21)$$

were computed to be

$$-0.065411390411077 \pm 0.18698774222604\sqrt{-1}$$

$$-4.179588609588923 \pm 2.428642661994181\sqrt{-1}$$

correct to the figures shown, without the use of a calculating machine. The computation required about six hours. The most difficult operation was found to be the extraction of the square root of p. For large numbers the ordinary root extraction process is extremely difficult. The labor was considerably reduced by using Horner's method to solve the equation $u^2 = p$. For a particular degree of accuracy decided upon in advance this method of root extraction becomes progressively easier.

A disadvantage of the method is that in general the roots are obtained to the same number of decimal places and, hence, to a different number of significant figures. This occurs when the real part of one root is somewhat smaller than that of the other. But in this situation an approximate solution by a method to be described later can be used to give a clue to the number of significant figures to carry through the solution in order to obtain a particular minimum number of significant figures in the result.

Type IV

The methods to be here described depend for their effectiveness upon there being some exceptional relation between the roots. By exceptional relation is meant, for example, two very nearly equal roots, or the most commonly encountered situation of the two absolute values being considerably different in magnitude. In only a few cases will it be possible to determine beforehand whether or not the roots are related in some way. In general a convenient assumption is made and an approximate solution obtained on that assumption. The approximate solution thus obtained is then corrected by trial and error or by one of the methods of correction to be described later.

The Method of Two Quadratics. Suppose that the absolute value of one root of a quartic is very much smaller than the absolute value of the other. Let r_1 be this small absolute value. Assume further that the real part u_1 is very much smaller than the other real part. In this situation equations (3) through (6) show that

$$A = -2u_2 \qquad B = r_2^2 \qquad C = -2u_1r_2^2 \qquad D = r_1^2r_2^2$$

to a first approximation. There does not seem to be any reliable criteria for determining whether or not a given equation is of this type. In fact, the only excuse for this method is that it does sometimes give fairly good results.

From the above expressions for the coefficients it appears that the roots of the quadratic equations

$$Bx^2 + Cx + D = 0 \qquad (u_1 \pm iv_1)$$

$$x^2 + Ax + B = 0 \qquad (u_2 \pm iv_2)$$

are approximately the roots of equation (1). The second quadratic should give a better approximation than the first. As an example, the roots of equation (21) obtained by this method are:

$$-0.06908 \pm 0.1807\sqrt{-1}$$

$$-4.245 \quad \pm 2.546\ \sqrt{-1}$$

The errors in the four numbers involved vary from 1.6% to 5.6% with a ratio between the absolute values of 0.041.

When this method fails because the absolute values do not differ greatly, one or two applications of Graeffe's root squaring process should separate the roots sufficiently to make the above method applicable.

Approximate solutions obtained as above can be improved by either of two methods to be described later.

Special Cases. There are other special relations between the roots that permit approximate solutions to be made easily. There are at least four such cases for which a relation exists among the coefficients of the quartic which may be used as a test for the existence of the relation. If the test equation is *nearly* satisfied by the coefficients of a given equation, then the roots may be found approximately by the formulae for that case. This follows from the fact that the roots of an algebraic equation are continuous functions of the coefficients. (See: Weber's "Algebra," Vol. I, Art. 44.)

Case 1. The given equation has a pure imaginary root. Setting $u_1 = 0$ in equations (3) through (6):

$$A = -2u_2 \qquad B = r_1^2 + r_2^2 \qquad C = -2r_1^2u_2 \qquad D = r_1^2r_2^2$$

Test equation:

$$A^2D - ABC + C^2 = 0$$

Solution:

$$r_1^2 = C/A \qquad r_2^2 = AD/C$$

$$u_2 = -A/2$$

$$v_1 = \pm r_1 \qquad v_2 = \pm\sqrt{r_2^2 - u_2^2}$$

Case 2. The real parts of the roots of the given equation are equal. Setting $u_1 = u_2 = u$ in equations (3) through (6):

$$A = -4u \qquad B = r_1^2 + 4u^2 + r_2^2 \qquad C = -2u(r_1^2 + r_2^2) \qquad D = r_1^2r_2^2$$

Test equation:

$$A(A^2 - 4B) + 8C = 0$$

Solution:

$$r^2 = \frac{C \pm \sqrt{C^2 - A^2D}}{A} = r_1^2 \text{ and } r_2^2$$

$$u_1 = u_2 = -A/4$$

$$v_1 = \pm\sqrt{r_1^2 - u_1^2} \qquad v_2 = \pm\sqrt{r_2^2 - u_2^2}$$

Case 3. The absolute values of the roots of the given equation are equal. Setting $r_1 = r_2 = r$ in equations (3) through (6):

$$A = -2(u_1 + u_2) \qquad C = -2r^2(u_1 + u_2)$$

$$B = 2(r^2 + 2u_1u_2) \qquad D = r^4$$

Test equation:

$$A^2D - C = 0$$

Solution:

$$u = -\tfrac{1}{4} \pm \tfrac{1}{4}\sqrt{A^2 - 4B + 8C/A} = u_1 \text{ and } u_2$$

$$r = +\sqrt{C/A}$$

$$v_1 = \pm\sqrt{r^2 - u_1^2} \qquad v_2 = \pm\sqrt{r^2 - u_2^2}$$

Case 4. The real parts of the roots are numerically equal but opposite in sign. Let the absolute value of the real parts be u. Then equations (3) through (6) give:

$$A = 0 \qquad B = r_1^2 - 4u^2 + r_2^2 \qquad C = 2u(r_1^2 - r_2^2) \qquad D = r_1^2r_2^2$$

In this case the only test is $A = 0$. There is no simple solution as in the other two cases, but the method of Type III is particularly easy to apply.

Use of the Special Cases. These special cases may be used in two ways, either as the basis of a method of approximate solution, or they may furnish an exact solution in a situation where another method fails.

A method of finding complex roots of algebraic equations devised by F. L. Hitchcock and published in the "Journal of Mathematics and Physics," June 1938, No. 2, Vol. XVII, is applicable, as developed by Dr. Hitchcock, to any algebraic equation; but when adapted especially for quartic equations, it becomes an application of Case 1.

The Method of F. L. Hitchcock. The roots of the given equation are reduced successively, by convenient increments, via Horner's method. The test quantity $(A^2D - ABC + C^2)$ is computed for each step from the coefficients of the resulting equation. A change of sign of this quantity between consecutive root reductions indicates that the real part of one root lies between the root reductions for those two steps. By interpolation and repeated trial, the interval including the change in sign may be made as small as desired; thus determining the real part

of the root to any desired degree of accuracy. The equation with reduced roots obtained at the last step may then be solved by the formulae of Case 1 to obtain the imaginary parts.

Because this method is quite general and is very efficient when three or four figure accuracy is desired an example in detail is in order. Equation (21) will be solved. In tabular form the work is as follows:

Roots reduced by:	Coefficients:				$A^2D - ABC + C^2$
	A	B	C	D	
0	+8.49	+24.50	+3.385	+0.917	−628
−1	+4.49	+5.03	−24.145	+14.542	+1421
−2	+0.49	−2.44	−24.735	+40.227	+593
−3	−3.51	+2.09	−22.385	+63.032	+1114
−4	−7.51	+18.62	−41.095	+92.017	+1140
−5	−11.51	+47.15	−104.865	+160.242	−24600
−4.1	−7.91	+20.93	−45.048	+96.320	+600
−4.2	−8.31	+23.37	−49.476	+101.042	−190
−4.18	−8.23	+22.87	−48.551	+100.062	−4

From the first six lines of the table it appears that the real parts of the roots lie between 0 and −1, and between −4 and −5. We choose to approximate the latter. By interpolation between +1140 and −24600, we obtain −4.1 for the next trial. The resulting +600 indicates −4.2 for the next trial. Then interpolation between +600 and −190 gives −4.18 as a closer approximation. For −4.18 the result that $A^2D - ABC + C^2 = -4$ shows the real part of this root to be very nearly −4.180, and we accept this as sufficiently accurate. From the formulae of Case 1:

$$r^2 = \frac{-48.55}{-8.230} = +5.899$$

$$v = \pm\sqrt{5.899} = \pm 2.429$$

So that the root is approximately:

$$-4.180 \pm 2.429\sqrt{-1}$$

This approximation can be improved by continuing the process used in finding −4.180 or, more easily by a special method of correction to be described later.

The foregoing method will prove troublesome if the real parts of the two roots are nearly equal. Hence before starting the solution it would be well to apply the test of Case 2. If $A(A^2 - 4B) + 8C$ is comparatively small the real parts are nearly equal and their mean value is $-A/4$.

It would seem that Case 3 would be used for a similar method of successive approximation but such a method would prove difficult for the test quantity $A^2D - C^2$ changes sign when $\cos^2 \phi_1$ and $\cos^2 \phi_2$ are equal as well as when the absolute values are equal.

A New Method. The method to be now described occurred to the writer during the study that led to the four special cases. It follows from the relations between the roots and the coefficients in equations (3) through (6). Consider equations (4) and (6):

$$B = r_1^2 + 4u_1u_2 + r_2^2 \tag{4}$$

$$D = r_1^2 r_2^2 \tag{6}$$

In most cases $4u_1u_2$ will be smaller numerically than $r_1^2 + r_2^2$, being negligible when either u_1 or u_2 is nearly zero. In any case when either u_1 or u_2 is comparatively small the equation

$$x^4 - Bx^2 + D = 0 \tag{22}$$

has roots that are fair approximations to r_1^2 and r_2^2, the closeness of the approximation depending on u_1 and u_2. Solving equations (3) and (5) simultaneously, we obtain

$$u_1 = -\frac{Ar_1^2 - C}{2(r_1^2 - r_2^2)} \qquad u_2 = +\frac{Ar_2^2 - C}{2(r_1^2 - r_2^2)} \tag{23}$$

If approximate values of r_1^2 and r_2^2 are found from (22) and substituted in (23), the resulting values of u_1 and u_2 will give an approximation to $4u_1u_2$ and this may then be subtracted from the given B to obtain a better approximation to $r_1^2 + r_2^2$; that is, we may now solve

$$x^4 - (B - 4u_1'u_2')x^2 + D = 0$$

where u_1' and u_2' are the values found from (23), to obtain a better approximation to r_1^2 and r_2^2. These may be used in (23) and a closer approximation to $4u_1u_2$ obtained, whence a better approximation to the coefficient of x^2; and so on to any desired degree of accuracy. The solution of equation (21) by this process is obtained quite easily as follows:

Coeff. of x^2	r_1^2	r_2^2	u_1	u_2
−24.50	0.0375	24.46	−0.0628	−4.14
−23.46	0.0390	23.42	−0.06531	−4.177
−23.409	0.03924	23.37	−0.06541	−4.179
−23.407	0.03924	23.368	−0.06541	−4.1796

$$v_1 = \pm\sqrt{0.03924 - 0.00428} = \pm 0.1870$$

$$v_2 = \pm\sqrt{23.368 - 17.469} = \pm 2.429$$

And the roots are:

$$-0.06541 \pm 0.1870\sqrt{-1}$$
$$-4.1796 \pm 2.429\sqrt{-1}$$

From equations (23) it is evident that this method will fail when the absolute values of the roots are equal, and will be rather inefficient if the absolute values are nearly equal. It is well, then, before starting a solution by this method, to apply the test of Special Case 3. If the test shows that the absolute values are nearly equal, the formulae of Case 3 will give a good approximation to the roots. The values of u_1 and u_2 found from these formulae may then be used in the process just described to obtain a better approximation.

Methods for Improving Approximate Roots. The last two methods of Type IV have the property that an improvement of an approximation can be made by a repetition of the process by means of which the approximation was derived. Either of these two methods may of course be used to improve an approximation found by other means. There are two other methods designed particularly for the improvement of an approximate root which merit some attention here.

Method of E. B. Wilson. In the publication "Theory of an Aeroplane Encountering Gusts," E. B. Wilson, S. Doc. 268, 64th Cong. 1st Sess., equation (21) is solved by the method of two quadratics and the approximate value

$$-4.245 \pm 2.545\sqrt{-1}$$

is found. This value is then corrected by an adaptation of Newton's method for real roots. That is, if r is an approximate root of $f(x) = 0$, then

$$x = r - \frac{f(r)}{f'(r)}$$

is a better approximation, where $f'(x)$ is the derivative of $f(x)$. The amount of computation needed in carrying out an improvement of an approximation by this method is considerable and the inherent weakness of Newton's method when multiple or nearly equal roots are present will in some cases lead to a loss of accuracy instead of an improvement of the approximation. Hence caution must be used in the application of this method and a check by actual substitution made to determine whether or not the approximate root has actually been improved.

Method of F. L. Hitchcock. The article by F. L. Hitchcock in the

"Journal of Mathematics and Physics," June 1938, No. 2, mentioned before describes a method of improving an approximation to a complex root of an equation of any degree. It is particularly efficient when used on quartic equations.

The procedure is briefly this: if $u + iv$ is an approximate root of $f(x) = 0$, we can reduce the roots of $f(x) = 0$, using Horner's method, by the amount u, to obtain an equation $g(x) = 0$ in which the root $u' + iv$, corresponding to $u + iv$, is nearly pure imaginary. That is, u' is a small quantity. Then from the first two terms of the Taylor's expansion for $g(u' + iv)$ we have:

$$g(u' + iv) = g(iv) + u'g'(iv) = 0$$

Whence:

$$u' = -\frac{g(iv)}{g'(iv)}$$

The quantity u' will, in general, be complex. It is the correction to be added to the approximate root $u + iv$. For ease in evaluating u' Dr. Hitchcock gives a set of formulas for the real and imaginary parts of $g(iv)$ and $g'(iv)$. They are:

$$\text{Re } g(iv) = D' - B'v^2 + v^4 \qquad \text{Re } g'(iv) = C' - 3A'v$$

$$\text{Im } g(iv) = v(C' - A'v^2) \qquad \text{Im } g'(iv) = 2v(B' - 2v)$$

Where A', B', C', and D' are the coefficients of x^3, x^2, x^1, and x^0, respectively, in the equation $g(x) = 0$.

As an example of this method, suppose equation (21) has been solved for the approximate root

$$-4.245 + 2.546\sqrt{-1}$$

Equation (21) with roots reduced by -4.245 is:

$$g(x) = x^4 - 8.490x^3 + 24.50x^2 - 51.63x + 103.3 = 0$$

The formulas with $v = 2.546$ then give:

$$u' = \frac{13.5 - 8.66\sqrt{-1}}{113 + 58.7\sqrt{-1}} = +0.063 - 0.109\sqrt{-1}$$

So that the corrected value of the root is:

$$-4.182 + 2.437\sqrt{-1}$$

which is correct to three significant figures. A repetition of the process will give four figures. The efficiency of the method lies in the fact that the correction u' is certain to be small and that the greater part of the computation is done by synthetic division, the remainder by slide rule.

A third method for the improvement of an approximate root is described in "Vorlesungen Uber Numerisches Rechnen," Runge und Kõnig, Art. 55, but it is of only academic interest for it gives an accurate correction only when the approximate root is already accurate beyond practical requirements.

ELLIPTIC AND SPHEROIDAL WAVE FUNCTIONS

By L. J. Chu and J. A. Stratton

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I. Introduction

1.1 Scalar Wave Equation. It is known that the wave equation[1]

[1] The most complete account of earlier researches on the subject matter of this paper is contained in a monograph by M. J. O. Strutt, "Lamésche, Mathieusche, und verwandte Funktionen in Physik und Technik," Springer, 1932. An extensive bibliography is included.

$$\nabla^2 W + k^2 W = 0 \tag{1}$$

may be separated in spheroidal coordinates ξ, η, φ obtained from the cylindrical coordinates r, φ, z through the transformation

$$r = f\sqrt{(\xi^2 \mp 1)(1 - \eta^2)}, \qquad \varphi = \varphi, \qquad z = f\xi\eta, \tag{2}$$

the upper or lower sign being chosen according as the coordinates are of prolate or oblate type. The factors of the resulting Lamé product are then

$$W = (\xi^2 \mp 1)^{\frac{m}{2}} U(\xi)(1 - \eta^2)^{\frac{m}{2}} V(\eta)\Phi(\varphi) \tag{3}$$

and satisfy the ordinary equations

$$\Phi''(\varphi) + m^2\Phi(\varphi) = 0, \tag{4}$$

$$(\xi^2 \mp 1)U'' + 2(m + 1)\xi U' - (b - f^2k^2\xi^2)U = 0, \tag{5}$$

$$(1 - \eta^2)V'' - 2(m + 1)\eta V' + (b \mp f^2k^2\eta^2)V = 0, \tag{6}$$

where b is a separation constant to be determined from conditions of finiteness and periodicity over the surface of a spheroid. Both (5) and (6) can be reduced to the standard form

$$(1 - z^2)U'' - 2(a + 1)zU' + (b - c^2z^2)U = 0. \tag{7}$$

The coordinates of the elliptic cylinder are expressed in terms of rectangular coordinates by the transformation

$$x = f\xi\eta, \qquad y = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}, \qquad z = z \tag{8}$$

In this system the scalar wave equation (1) separates into the factors

$$W = U(\xi)V(\eta)Z(z) \tag{9}$$

satisfying the ordinary equations

$$Z'' + \gamma^2 Z = 0, \tag{10}$$

$$(\xi^2 - 1)U'' + \xi U' - [b - f^2(k^2 - \gamma^2)\xi^2]U = 0 \tag{11}$$

$$(1 - \eta^2)V'' - \eta V' + [b - f^2(k^2 - \gamma^2)\eta^2]V = 0. \tag{12}$$

Equations (11) and (12) are obviously special cases of (7) for $a = -\frac{1}{2}$. Their solutions are known generally as Mathieu functions.

To visualize the geometrical significance of certain limiting cases the following will prove helpful. The eccentricity of an ellipse ξ = constant is $e = 1/\xi$. Let l be the length of a semi-major axis. Then the coordinate ξ of an ellipse is

$$\xi = l/f = 1/e, \tag{13}$$

thus appearing to be a pure number, the ratio of semi-major axis to semi-focal distance, or reciprocal of the eccentricity. The limiting case of a circle occurs when $f \to 0$, $\xi \to \infty$ such that the product ξf remains finite and equal to the limiting radius l. Identical relations hold for the spheroids obtained by rotating an ellipse about a principal axis.

1.2 **Vector Wave Equation.** In the theory of elasticity, as well as in electromagnetic theory, one must deal frequently with vector rather than scalar wave functions. Thus the equation governing the propagation of elastic waves in a homogeneous, isotropic medium is

$$\nabla \times \nabla \times \mathbf{s} - (\lambda + 2\mu)\nabla\nabla \cdot \mathbf{s} + \rho \frac{\partial^2 \mathbf{s}}{\partial t^2} = 0, \tag{14}$$

where $\mathbf{s}$ is the relative displacement of a point in the medium, ρ is the density and λ and μ are elastic constants. Likewise an electromagnetic field whose angular frequency is ω can be derived from the Hertz vectors $\mathbf{\Pi}$ and $\mathbf{\Pi}^*$ according to the rules[2]

$$\mathbf{E} = k^2\mathbf{\Pi} + \nabla\psi - i\omega\mu\nabla \times \mathbf{\Pi}^* \tag{15}$$

$$\mathbf{H} = (i\omega\epsilon + \sigma)\nabla \times \mathbf{\Pi} + k^2\mathbf{\Pi}^* + \nabla\psi^*, \tag{16}$$

$$k^2 = \mu\epsilon\omega^2 - i\mu\sigma\omega, \tag{17}$$

where μ, ϵ, σ are respectively permeability, dielectric constant, and conductivity in rationalized m.k.s. units. The electric and magnetic Hertzian vectors $\mathbf{\Pi}$ and $\mathbf{\Pi}^*$ are independent and satisfy

$$-\nabla \times \nabla \times \mathbf{\Pi} + k^2\mathbf{\Pi} + \nabla\psi = 0. \tag{18}$$

The scalars ψ and ψ^* are likewise independent. Usually they are placed equal to the divergence of the corresponding Hertz vector, but any solution of a homogeneous scalar wave equation may be employed.

Solutions of vector equations such as (14) and (18) can be constructed from solutions of the scalar wave equation.[3] Thus any rectangular component of a Hertzian vector satisfies (1). The differential operators may be expressed in terms of curvilinear coordinates and upon carrying out the operations indicated in (15) and (16) an electromagnetic field is obtained in a curvilinear system of coordinates. Such fields must

[2] J. A. Stratton, "Electromagnetic Theory," pp. 28–32, McGraw-Hill, 1941.
[3] J. A. Stratton, loc. cit., pp. 349–354, 392–399, 414–420.

eventually be superposed to satisfy boundary conditions on a prescribed surface. In place of a rectangular component it may be shown that the spherical function Π_R/R is also a solution of the scalar wave equation, where Π_R is a radial component. The field derived from a single component Π_R has no radial component of $\mathbf{H}$; that derived from Π_R^* has no radial component of $\mathbf{E}$. Unfortunately a vector field satisfying boundary conditions on a spheroid cannot be obtained in the same way from Hertz vectors which are everywhere normal to the surface of the spheroid. Such fields must be constructed by superposition of partial fields derived in the manner just described.

1.3 **Gegenbauer Functions.** The standard equation (7) is characterized by regular singularities at $z = \pm 1$ and an irregular point at infinity. The object of this investigation is to obtain independent solutions in the neighborhood of each singularity reduced to a form suitable for numerical calculations and to discuss their analytic continuation over the entire complex plane of z.

From (13) we note that if $f \to 0$, $\xi \to \infty$ in such a way that $f\xi \to R$, the spheroid reduces to a sphere of radius R. In this limit (7) is replaced by

$$z^2U'' + 2(a + 1)zU' + (c^2z^2 - b)U = 0, \tag{19}$$

an equation satisfied by half-order Bessel functions. For this reason the spheroidal functions will be represented in the neighborhood of the essential singularity at infinity by expansions in Bessel functions.

If, on the other hand, we place $c = 0$, (7) reduces to

$$(1 - z^2)U'' - 2(a + 1)zU' + bU = 0. \tag{20}$$

This is essentially the Gegenbauer equation, a hypergeometric equation with three regular singularities.[4] Since the spheroidal functions will be expressed in terms of the solutions of (20), we shall give a brief account of their properties.

The functions $T_n^a(z)$ and $\mathfrak{T}_n^a(z)$ are defined for all values of a, n, and z by

$$T_n^a(z) = (z^2 - 1)^{-a/2}P_{n+a}^a(z), \tag{21}$$

$$\mathfrak{T}_n^a(z) = (z^2 - 1)^{-a/2}Q_{n+a}^a(z), \tag{22}$$

where $P_{n+a}^a(z)$ and $Q_{n+a}^a(z)$ are the associated Legendre functions of the

[4] Whitaker and Watson, "Modern Analysis," p. 329, 4th ed., Cambridge University Press.

first and second kinds. Definitions of the Legendre functions differ in the literature with regard to phase, or the location of the cut in the z-plane. Accordingly, we shall adhere to the definitions given by Hobson.[5]

$$P_n^a(z) = \frac{e^{-n\pi i}(n+a)!}{4\pi \sin n\pi\, 2^n n!}(z^2-1)^{a/2} \times \int_C^{(z+,1+,z-,1-)} (t^2-1)^n (t-z)^{-n-a-1}\,dt \quad (23)$$

$$Q_n^a(z) = \frac{i\,e^{-n\pi i}(n+a)!}{4 \sin n\pi\, 2^n n!}(z^2-1)^{a/2} \times \int_C^{(-1+,1-)} (t^2-1)^n (t-z)^{-n-a-1}\,dt. \quad (24)$$

The functions T_n^a and $\mathfrak{T}_n^a$ differ slightly from those defined originally by Gegenbauer. They satisfy the equation

$$(z^2-1)w'' + 2(a+1)zw' - n(n+2a+1)w = 0. \quad (25)$$

But (25) is satisfied likewise by each of the following eight functions:

$$T_n^a,\quad T_{-n-2a-1}^a,\quad (z^2-1)^{-a}T_{n+2a}^{-a},\quad (z^2-1)^{-a}T_{-n-1}^{-a},$$

$$\mathfrak{T}_n^a,\quad \mathfrak{T}_{-n-2a-1}^a,\quad (z^2-1)^{-a}\mathfrak{T}_{n+2a}^{-a},\quad (z^2-1)^{-a}\mathfrak{T}_{-n-1}^{-a}.$$

Since the second order equation (25) admits only two independent solutions, certain linear relations must exist between the eight functions above. These follow directly from the theory of the hypergeometric function, and they are readily deduced from corresponding expressions for the Legendre functions given by Hobson.

$$T_n^a = T_{-n-2a-1}^a \quad (26)$$

$$T_{n+2a}^{-a} = T_{-n-1}^{-a} \quad (27)$$

$$\mathfrak{T}_a^n = e^{2a\pi i}\frac{(n+2a)!}{n!}(z^2-1)^{-a}\mathfrak{T}_{n+2a}^{-a} \quad (28)$$

$$\mathfrak{T}_{-n-2a-1}^a = e^{2a\pi i}\frac{(-n-1)!}{(-n-2a-1)!}(z^2-1)^{-a}\mathfrak{T}_{-n-1}^{-a} \quad (29)$$

$$T_n^a = \frac{e^{-a\pi i}}{\pi \cos (n+a)\pi}[\sin (n+2a)\pi\,\mathfrak{T}_n^a - \sin n\pi\,\mathfrak{T}_{-n-2a-1}^a] \quad (30)$$

[5] E. W. Hobson, "Spheroidal and Ellipsoidal Harmonics," pp. 188 and 195, Cambridge University Press, 1931.

$$T^{-a}_{n+2a} = \frac{e^{a\pi i}}{\pi \cos (n+a)\pi}[\sin n\pi\, \mathfrak{T}^{-a}_{n+2a} - \sin (n+2a)\pi\, \mathfrak{T}^{-a}_{-n-1}] \tag{31}$$

$$T^{a}_{n} = \frac{e^{a\pi i}(n+2a)!\sin (n+2a)\pi}{\pi n! \cos (n+a)\pi}(z^2-1)^{-a}[\mathfrak{T}^{-a}_{n+2a} - \mathfrak{T}^{-a}_{-n-1}] \tag{32}$$

$$(z^2-1)^{-a}T^{-a}_{n+2a} = \frac{e^{-a\pi i} n! \sin n\pi}{\pi(n+2a)!\cos (n+a)\pi}[\mathfrak{T}^{a}_{n} - \mathfrak{T}^{a}_{-n-2a-1}] \tag{33}$$

$$\frac{2}{\pi}e^{-a\pi i}\sin a\pi\, \mathfrak{T}^{a}_{n} = T^{a}_{n} - \frac{(n+2a)!}{n!}(z^2-1)^{-a}T^{-a}_{n+2a} \tag{34}$$

$$\frac{2}{\pi}e^{-a\pi i}\sin a\pi\, \mathfrak{T}^{a}_{-n-2a-1} = T^{a}_{n} - \frac{(-n-1)!}{(-n-2a-1)!}(z^2-1)^{-a}T^{-a}_{n+2a} \tag{35}$$

The factorial has been employed in place of the gamma function for non-integral as well as integral values of the argument, so that $a! = \Gamma(a+1)$.

In addition to these formulas connecting the various solutions of (25) frequent use will be made of the recurrence relations

$$z\,T^{a}_{n} = \frac{1}{2n+2a+1}[(n+1)T^{a}_{n+1} + (n+2a)\,T^{a}_{n-1}] \tag{36}$$

$$(z^2-1)\frac{d}{dz}T^{a}_{n} = (n+1)\,T^{a}_{n+1} - (n+2a+1)z\,T^{a}_{n} \tag{37}$$

which apply also to $\mathfrak{T}^{a}_{n}$.

From (21) and (22) and the known properties of Legendre functions the solutions of (25) can be expressed in terms of hypergeometric functions. Thus for $|1-z|<2$,

$$T^{a}_{n} = \frac{1}{(-a)!}(z-1)^{-a}F\left(-n-a, n+a+1; 1-a; \frac{1-z}{2}\right), \tag{38}$$

while for $|z|>1$

$$\mathfrak{T}^{a}_{n} = \frac{e^{a\pi i}2^{a}\sqrt{\pi}\,(n+2a)!}{(n+a+\tfrac{1}{2})!\,(2z)^{n+2a+1}} \times F\left(\frac{n+2a+2}{2}, \frac{n+2a+1}{2}, n+a+\frac{3}{2}, \frac{1}{z^2}\right) \tag{39}$$

where

$$F(a,b;c;z) = \frac{(c-1)!}{(a-1)!\,(b-1)!}\sum_{s=0}^{\infty}\frac{(a+s-1)!\,(b+s-1)!}{s!\,(c+s-1)!}z^{s}. \tag{40}$$

Through the transformations of the hypergeometric function and the relations (26)–(35) one may establish series expansions for the functions T and $\mathfrak{T}$ for the entire complex domain of z.

1.4 **Gegenbauer Functions for Integer Values of a.** Spheroidal wave functions associated with a complete spheroidal surface must be periodic in the equatorial angle φ. In this case a is a positive integer or zero and will be designated by the letter m. The relations between the functions defined in Sec. 1.3 then reduce as follows. For all values of n we have

$$T_n^m = T_{-n-2m-1}^m \tag{41}$$

$$T_{n+2m}^{-m} = T_{-n-1}^{-m} \tag{42}$$

$$T_n^m = \frac{(n+2m)!}{n!}(z^2-1)^{-m} T_{n+2m}^{-m} \tag{43}$$

$$\mathfrak{T}_n^m = \frac{(n+2m)!}{n!}(z^2-1)^{-m} \mathfrak{T}_{n+2m}^{-m} \tag{44}$$

$$\mathfrak{T}_{-n-2m-1}^m = \frac{(n+2m)!}{n!}(z^2-1)^{-m} \mathfrak{T}_{-n-1}^{-m} \tag{45}$$

$$T_n^m = \frac{\tan n\pi}{\pi}(\mathfrak{T}_n^m - \mathfrak{T}_{-n-2m-1}^m) \tag{46}$$

Numerical values of these functions in various regions of the z-plane may be obtained from the following hypergeometric series:

For $|z| > 1$,

$$\begin{aligned}\mathfrak{T}_n^{\pm m} &= \frac{(-2)^{\mp m}\sqrt{\pi}\,(n \pm 2m)!}{(n \pm m + \tfrac{1}{2})!\,(2z)^{n+1}} \\ &\quad \times (z^2-1)^{\mp m} F\left(\frac{n+2}{2}, \frac{n+1}{2}; n \pm m + \frac{3}{2}; \frac{1}{z^2}\right) \\ &= \frac{(-2)^{\pm m}\sqrt{\pi}\,(n \pm 2m)!}{(n \pm m + \tfrac{1}{2})!\,(2z)^{n\pm 2m+1}} \\ &\quad \times F\left(\frac{n \pm 2m + 2}{2}, \frac{n \pm 2m + 1}{2}; n \pm m + \frac{3}{2}; \frac{1}{z^2}\right)\end{aligned} \tag{47}$$

For $|1 - z| < 2$,

$$T_n^m = \frac{(n+2m)!}{2^m n!\, m!} F\left(-n, n+2m+1; 1+m; \frac{1-z}{2}\right), \tag{48}$$

$$\mathfrak{T}_n^m = \tfrac{1}{2}\, T_n^m\left[\ln\frac{z+1}{z-1} - \sigma(n) - \sigma(n+2m)\right] - \frac{\sin n\pi}{2\pi}\left[(z-1)^{-m}\right.$$
$$\times \sum_{r=0}^{m-1} \frac{(-1)^r(-n-m+r-1)!\,(n+m+r)!\,(m-r-1)!}{r!}\left(\frac{1-z}{2}\right)^r$$
$$+ (-2)^{-m} \sum_{r=0}^{\infty} \frac{(-n+r-1)!\,(n+2m+r)!}{r!\,(m+r)!}\,\sigma(r)\left(\frac{1-z}{2}\right)^r \tag{49}$$
$$+ (-1)^m \frac{(n+2m)!}{n!}\,(z+1)^{-m}$$
$$\left.\times \sum_{r=0}^{\infty} \frac{(-n-m+r-1)!\,(n+m+r)!}{r!\,(r+m)!}\,\sigma(m+r)\left(\frac{1-z}{2}\right)^r\right]$$

wherein $n \geqq 0$ and

$$\sigma(x) = \frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{x}. \tag{50}$$

For $|z| < 1$ and $Im(z) \neq 0$,

$$\mathfrak{T}_n^m = \frac{i^{\mp(1+n)}\sqrt{\pi}\left(\dfrac{n+2m-1}{2}\right)!}{2^{1-m}\left(\dfrac{n}{2}\right)!}\,F\left(\frac{n+2m+1}{2}, -\frac{n}{2}; \frac{1}{2}; z^2\right)$$
$$+ \frac{i^{\mp n}\sqrt{\pi}\left(\dfrac{n+2m}{2}\right)!\,2^m}{\left(\dfrac{n-1}{2}\right)!}\,z\,F\left(\frac{n+2m+2}{2}, \frac{1-n}{2}; \frac{3}{2}; z^2\right), \tag{51}$$

$$T_n^m = 2^m \cos\frac{n\pi}{2}\,\frac{\left(\dfrac{n+2m-1}{2}\right)!}{\sqrt{\pi}\left(\dfrac{n}{2}\right)!}\,F\left(\frac{n+2m+1}{2}, \frac{-n}{2}; \frac{1}{2}; z^2\right)$$
$$+ 2^{m+1}\sin\frac{n\pi}{2}\,\frac{\left(\dfrac{n+2m}{2}\right)!}{\sqrt{\pi}\left(\dfrac{n-1}{2}\right)!}\,z\,F\left(\frac{n+2m+2}{2}, \frac{1-n}{2}; \frac{3}{2}; z^2\right) \tag{52}$$

Usually we shall have to deal with functions for which n as well as m is a positive integer. In this case some of the above series break off and hence are finite for all finite values of z. Thus when both n and m

are positive integers the various solutions fall into two classes: a set of integral functions, and a set characterized by singularities at $z = \pm 1$ as well as at infinity.

TABLE I

n and m positive integers

T_n^m	$(z^2-1)^{-m}T_{n+2m}^{-m}$	no singularities in the z-plane other than at $z = \infty$
$T_{-n-2m-1}^m$	$(z^2-1)^{-m}T_{-n-1}^{-m}$	
$\mathfrak{T}_{-n-2m-1}^m$	$(z^2-1)^{-m}\mathfrak{T}_{-n-1}^{-m}$	
$\mathfrak{T}_n^m$	$(z^2-1)^{-m}\mathfrak{T}_{n+2m}^{-m}$	singularities at $z = \pm 1$ and $z = \infty$

If n is a positive or negative integer $\mathfrak{T}_n^m$ becomes infinite when $n \leqq -2m - 1$ in virtue of the factor $(n + 2m)!$ in the numerator. It is finite if $n > -2m - 1$. Likewise T_n^m is zero when $-2m - 1 < n < 0$. It is finite for all other values of n.

1.5 **Gegenbauer Functions for $a = -\frac{1}{2}$.** Comparable in importance to the case $a = m$ is that in which $a = -\frac{1}{2}$. We have seen that this leads to the Mathieu functions. If we let $z = \cosh \psi$, Eq. (20) reduces for $a = -\frac{1}{2}$ to

$$\frac{d^2U}{d\psi^2} - bU = 0 \tag{53}$$

satisfied by hyperbolic functions. From the definitions of T_n^a and $\mathfrak{T}_n^a$ it can be shown that

$$\mathfrak{T}_n^{-\frac{1}{2}} = -\frac{i}{n}\sqrt{\frac{\pi}{2}}\,e^{-n\psi} \tag{54}$$

$$(z^2-1)^{\frac{1}{2}}\mathfrak{T}_{n-1}^{\frac{1}{2}} = i\sqrt{\frac{\pi}{2}}\,e^{-n\psi} \tag{55}$$

$$T_0^{-\frac{1}{2}} = \sqrt{\frac{2}{\pi}}\,\psi \tag{56}$$

$$T_n^{-\frac{1}{2}} = \frac{1}{n}\sqrt{\frac{2}{\pi}}\sinh n\psi \tag{57}$$

$$(z^2-1)^{\frac{1}{2}}T_{n-1}^{\frac{1}{2}} = \sqrt{\frac{2}{\pi}}\cosh n\psi \tag{58}$$

for all values of n. The relations between the functions reduce to

$$T_n^{-1} = T_{-n}^{1} \tag{59}$$

$$\mathfrak{T}_n^{-1} = -\frac{1}{n}(z^2 - 1)^{\frac{1}{2}}\mathfrak{T}_{n-1}^{1} \tag{60}$$

$$T_n^{-1} = -\frac{i}{\pi}(\mathfrak{T}_n^{-1} + \mathfrak{T}_n^{1}) \tag{61}$$

$$(z^2 - 1)^{\frac{1}{2}}T_{n-1}^{1} = \frac{in}{\pi}(\mathfrak{T}_n^{-1} - \mathfrak{T}_n^{1}) \tag{62}$$

$$\mathfrak{T}_n^{-1} = \frac{i\pi}{2}\left[T_n^{-1} - \frac{1}{n}(z^2 - 1)^{\frac{1}{2}}T_{n-1}^{1}\right] \tag{63}$$

These are all integral functions, having no singularities other than at $z = \infty$.

II. General Spheroidal Functions

2.1 **Definition.** In Sec. 1.3 it was noted that Eq. (7) reduces to (19) when $z \gg 1$ and $c \to 0$, and is then satisfied by

$$U = (cz)^{-a-\frac{1}{2}}Z_{l+a+\frac{1}{2}}(cz) \tag{64}$$

where $Z_p(cz)$ is any Bessel function of order p and argument cz. Likewise when $c \to 0$ Eq. (7) reduces to (20) and is satisfied by T_l^a or $\mathfrak{T}_l^a$ provided b has the characteristic value $l(l + 2a + 1)$. In general we shall express b in the form

$$b = l(l + 2a + 1) + \epsilon_l(a, c) \tag{65}$$

where ϵ_l is a function of the parameters l, a and c, and vanishes with c. A circle drawn about the origin as a center and passing through the singular points at $z = \pm 1$ divides the z-plane into two domains and limits the convergence of most series representations. We consider two systems of solutions, Those solutions of (7) that are valid in the neighborhood of the essential singularity at infinity we designate by U. They shall be represented in terms of the Bessel functions (64). Those solutions about the ordinary point $z = 0$ and converging at least within a circle of unit radius are to be designated by V, and will be represented by expansions in Gegenbauer functions. These representations are remarkable in that only one set of expansion coefficients need be determined. The problems of numerical computations and of determining the connections linking the solutions in one domain with those in another are thus vastly simplified.

2.2 U-Functions. The solutions of (7) valid when $z \gg 1$ will be defined by the expansion

$$U(a, c; z) = (cz)^{-a-\frac{1}{2}} \sum_n a_n Z_{n+\rho+a+\frac{1}{2}}(cz), \tag{66}$$

where n is an integer and $0 \leqq \rho < 1$. Upon introducing this series into (7) and applying the recurrence relations for Bessel functions it is found that the coefficients a_n satisfy the three term recursion formula

$$\begin{aligned} &\frac{(n+\rho+1)(n+\rho+2)}{(2n+2\rho+2a+3)(2n+2\rho+2a+5)}\, a_{n+2} \\ &\quad + \frac{(n+\rho+2a-1)(n+\rho+2a)}{(2n+2\rho+2a-3)(2n+2\rho+2a-1)}\, a_{n-2} \\ &\quad + \left[\frac{b-(n+\rho)(n+\rho+2a+1)}{c^2}\right. \\ &\quad \left. - \frac{2(n+\rho)^2+2(n+\rho)(2a+1)+2a-1}{(2n+2\rho+2a-1)(2n+2\rho+2a+3)}\right] a_n = 0. \end{aligned} \tag{67}$$

When $c \to 0$ there remains in the limit only

$$[b-(n+\rho)(n+\rho+2a+1)]a_n = 0.$$

If then we choose

$$b = (l+\rho)(l+\rho+2a+1),$$

where l is either an even or an odd integer, it is apparent that

$$a_n \to 0, \quad \text{for} \quad n \gtrless l$$

and U reduces to a single term of order $l+\rho+a+\frac{1}{2}$. When c differs from zero the summation extends from $-\infty$ to $+\infty$. As the subscripts of a_n in (67) differ always by an even number, the even and odd series are distinct: the summation extends over even values of n when l is even, over odd values when l is odd.

In order that (66) shall be an analytic representation of a solution it is necessary that the series converge for $cz > 1$. The maximum absolute magnitude of a Bessel function is independent of its order when $cz \to \infty$. The coefficients a_n must therefore converge in both directions from a_l, the dominant term. This condition, in turn, restricts the separation constant b to a set of characteristic values b_l.

The quantity ρ plays a role analogous to the roots of an indicial equation in the power series solution of a differential equation. By proper

choice of ρ it is possible to terminate a certain number of the expansions at $n = 0$ and thus to restrict the summation to positive values of n. However, this is untrue in other cases and in general the summation must be extended from $-\infty$ to $+\infty$, over either even or odd values of n. An earlier paper[6] failed to take this into account and the conclusions drawn there must be modified accordingly.

If now we let

$$U(a, c; z) = (z^2 - 1)^{-s} X(a, c, z), \tag{68}$$

it is apparent that X must satisfy the equation

$$(z^2 - 1)X'' + 2(1 - a)zX' + (c^2z^2 - b - 2a)X = 0 \tag{69}$$

and X can be represented by expansions of the form

$$X(a, c; z) = (cz)^{s-1} \sum_{n'}' a'_{n'} Z_{n'-s+1}(cz), \tag{70}$$

where the $a'_{n'}$ satisfy the recursion formula

$$\begin{aligned} &\frac{(n' + 1)(n' + 2)}{(2n' - 2a + 3)(2n' - 2a + 5)} a'_{n'+2} \\ &\quad + \frac{(n' - 2a - 1)(n' - 2a)}{(2n' - 2a - 3)(2n' - 2a - 1)} a'_{n'-2} \\ &\quad + \left[\frac{b + 2a - (n' - 2a + 1)n'}{c^2}\right. \\ &\quad \left. - \frac{2n'^2 + 2n'(1 - 2a) - 2a - 1}{(2n' - 2a + 3)(2n' - 2a - 1)}\right] a'_{n'} = 0. \end{aligned} \tag{71}$$

Upon comparing (71) with the recursion formula (67) we observe that the two are identical if

$$\begin{aligned} n' &= n + \rho + 2a, \\ a'_{n'} &= \frac{(n + \rho)!}{(n + \rho + 2a)!} Ca_n, \end{aligned} \tag{72}$$

where C is any constant. Since the Bessel functions of negative order satisfy the same relations as those of positive order, further solutions of (7) are obtained involving the same coefficients a_n.

We shall define a set of eight spheroidal functions as follows in terms of the Bessel functions of the first and second kinds. The normalization of the coefficients will be discussed later.

$$U_1(a, c; z) = (cz)^{-s-1} \sum_{n}' a_n J_{n+\rho+s+1}(cz) \tag{73}$$

[6] J. A. Stratton, Proc. Nat. Acad. Sci., *6*, 51–56, 1935 and *6*, 316–321, 1935.

$$U_2(a, c; z) = (cz)^{-\alpha-\frac{1}{2}} \sum_n{}' a_n J_{-n-\rho-\alpha-\frac{1}{2}}(cz) \tag{74}$$

$$U_3(a, c; z) = (cz)^{\alpha-\frac{1}{2}}(z^2-1)^{-\alpha} \sum_n{}' a_n \frac{(n+\rho)!}{(n+\rho+2a)!} J_{n+\rho+\alpha+\frac{1}{2}}(cz) \tag{75}$$

$$U_4(a, c; z) = (cz)^{\alpha-\frac{1}{2}}(z^2-1)^{-\alpha} \sum_n{}' a_n \frac{(n+\rho)!}{(n+\rho+2a)!} J_{-n-\rho-\alpha-\frac{1}{2}}(cz) \tag{76}$$

$$U_5(a, c; z) = (cz)^{-\alpha-\frac{1}{2}} \sum_n{}' a_n N_{n+\rho+\alpha+\frac{1}{2}}(cz) \tag{77}$$

$$U_6(a, c; z) = (cz)^{-\alpha-\frac{1}{2}} \sum_n{}' a_n N_{-n-\rho-\alpha-\frac{1}{2}}(cz) \tag{78}$$

$$U_7(a, c; z) = (cz)^{\alpha-\frac{1}{2}}(z^2-1)^{-\alpha} \sum_n{}' a_n \frac{(n+\rho)!}{(n+\rho+2a)!} N_{n+\rho+\alpha+\frac{1}{2}}(cz) \tag{79}$$

$$U_8(a, c; z) = (cz)^{\alpha-\frac{1}{2}}(z^2-1)^{-\alpha} \sum_n{}' a_n \frac{(n+\rho)!}{(n+\rho+2a)!} N_{-n-\rho-\alpha-\frac{1}{2}}(cz). \tag{80}$$

The coefficients a_n of all these solutions satisfy the recursion formula (67), and summation is extended from negative infinity to positive infinity. The prime over the summation sign indicates that only even values of n are to be taken if the index l of the separation constant is even, only odd values if l is odd.

2.3 **V-Functions.** We construct next a set of solutions for the domain $|z| < 1$. When $c = 0$ Eq. (7) is satisfied by $T^a_{l+\rho}(z)$ provided that the characteristic value b_l belongs to the set $(l+\rho)(l+\rho+2a+1)$. If, therefore, c does not vanish, one is led to try solutions of the form

$$V(a, c; z) = \sum_n{}' d_n T^a_{n+\rho}(z), \tag{81}$$

where n is an odd or even integer and $0 \leqq \rho < 1$. The l^{th} term of the series is dominant when c is small. Upon substitution of V into (7) and applying the differential and recurrence relations of the Gegenbauer functions it is found that the coefficients d_n satisfy the recursion formula

$$\begin{aligned} &\frac{(n+\rho+2a+2)(n+\rho+2a+1)}{(2n+2\rho+2a+5)(2n+2\rho+2a+3)} d_{n+2} \\ &\quad + \frac{(n+\rho)(n+\rho-1)}{(2n+2\rho+2a-1)(2n+2\rho+2a-3)} d_{n-2} \\ &\quad + \left[\frac{2(n+\rho)^2 + 2(n+\rho)(2a+1) + 2a - 1}{(2n+2\rho+2a-1)(2n+2\rho+2a+3)}\right. \\ &\quad \left. + \frac{(n+\rho)(n+\rho+2a+1) - b}{c^2}\right] d_n = 0. \end{aligned} \tag{82}$$

By comparison of (82) and (67) it is easily shown that, apart from a proportionality constant to be determined by the normalization,

$$d_n = i^n \frac{(n+\rho)!}{(n+\rho+2a)!} a_n. \tag{83}$$

The recursion formula (82) is unmodified when $n + \rho$ is replaced by $-n - \rho - 2a - 1$, so that two more solutions can be constructed from the functions $T^a_{-n-\rho-2a-1}$, $\mathfrak{T}^a_{-n-\rho-2a-1}$. Furthermore, as in the case of the U-functions, another four solutions are obtained by the transformation

$$V(a, c; z) = (z^2 - 1)^{-a} Y(a, c; z), \tag{84}$$

where $Y(a, c; z)$ satisfies (69). Thus eight formal solutions of (7) are established which we designate as follows:

$$V_1(a, c; z) = (z^2 - 1)^{-a} \sum_n{}' a_n i^n \mathfrak{T}^a_{-n-\rho-1}(z) \tag{85}$$

$$V_2(a, c; z) = (z^2 - 1)^{-a} \sum_n{}' a_n i^n \mathfrak{T}^a_{n+\rho+2a}(z) \tag{86}$$

$$V_3(a, c; z) = \sum_n{}' a_n i^n \frac{(n+\rho)!}{(n+\rho+2a)!} \mathfrak{T}^a_{-n-\rho-2a-1}(z) \tag{87}$$

$$V_4(a, c; z) = \sum_n{}' a_n i^n \frac{(n+\rho)!}{(n+\rho+2a)!} \mathfrak{T}^a_{n+\rho}(z) \tag{88}$$

$$V_5(a, c; z) = (z^2 - 1)^{-a} \sum_n{}' a_n i^n T^a_{-n-\rho-1}(z) \tag{89}$$

$$V_6(a, c; z) = (z^2 - 1)^{-a} \sum_n{}' a_n i^n T^a_{n+\rho+2a}(z) \tag{90}$$

$$V_7(a, c; z) = \sum_n{}' a_n i^n \frac{(n+\rho)!}{(n+\rho+2a)!} T^a_{-n-\rho-2a-1}(z) \tag{91}$$

$$V_8(a, c; z) = \sum_n{}' a_n i^n \frac{(n+\rho)!}{(n+\rho+2a)!} T^a_{n+\rho}(z) \tag{92}$$

Again the prime over the summation sign indicates that odd or even integer values of n are to be taken according as l is odd or even. Unless otherwise specified the summation is from negative to positive infinity.

For large values of $|n|$ the absolute magnitude of the ratio T^a_n/T^a_{n+2} or $\mathfrak{T}^a_n/\mathfrak{T}^a_{n+2}$ approaches unity at either $z = 0$ or $z = \pm 1$. Thus the convergence of the V-functions, as was that of the U-functions, is determined by the convergence of the sum $\sum_{n=-\infty}^{\infty}{}' a_n$.

2.4 Calculation of the Coefficients a_n and the Separation Constant. For sufficiently small values of c one may express the separation constant b and the coefficients a_n as power series in c. These series are then substituted in the recursion formula (67). Upon collecting terms and equating coefficients of like powers of c to zero one obtains in a single process both the characteristic values b and the expansion coefficients a_n. However, the convergence of these expressions is usually slow and they prove unsatisfactory for values of $c^2 > 1$. At present the most useful treatment of three term recursion formulas is based on a method of continued fractions.[7] We shall describe briefly the procedure.

To avoid cumbersome factors let us write the fundamental recursion formula (67) in the form

$$A_{n+2}a_{n+2} + B_n a_n + C_{n-2}a_{n-2} = 0 \tag{93}$$

where

$$A_n = \frac{(n+\rho)(n+\rho-1)}{(2n+2\rho+2a-1)(2n+2\rho+2a+1)},$$

$$B_n = \frac{b-(n+\rho)(n+\rho+2a+1)}{c^2} - \frac{2(n+\rho)^2+2(n+\rho)(2a+1)+2a-1}{(2n+2\rho+2a-1)(2n+2\rho+2a+3)}, \tag{94}$$

$$C_n = \frac{(n+\rho+2a+1)(n+\rho+2a+2)}{(2n+2\rho+2a+1)(2n+2\rho+2a+3)}.$$

Evidently (93) can be developed as a continued fraction. Thus the ratio a_l/a_{l-2} is

$$\frac{a_l}{a_{l-2}} = -\frac{C_{l-2}}{B_l + A_{l+2}\dfrac{a_{l+2}}{a_l}} = -\cfrac{C_{l-2}}{B_l - \cfrac{A_{l+2}C_l}{B_{l+2} - \ddots - \cfrac{A_n C_{n-2}}{B_n + A_{n+2}\dfrac{a_{n+2}}{a_n}}}}. \tag{95}$$

[7] E. L. Ince, Phil. Mag., *6*, 547–558, 1928.

If the series is to converge, the value of a_l/a_{l-2} will be unaffected by the last term a_{n+2}/a_n provided n is chosen sufficiently large. In practice one may usually start with n ten or twelve integers larger than l, neglect a_{l+12}/a_{l+10}, and upon running up the fraction an expression for the initial ratio a_l/a_{l-2} is obtained in terms of b which is contained explicitly in B. Repetition of this process with a neighboring value of n will indicate whether or not a sufficient number of terms have been taken into account.

Next the ratio a_l/a_{l-2} is approached in the opposite direction. Eq. (93) can be developed in the form

$$\frac{a_{l-2}}{a_l} = -\frac{A_l}{B_{l-2} + C_{l-4}\dfrac{a_{l-4}}{a_{l-2}}}$$

$$= -\cfrac{A_l}{B_{l-2} - \cfrac{A_{l-2}C_{l-4}}{B_{l-4} - \ddots - \cfrac{A_{-n+2}C_{-n}}{B_{-n} + C_{-n-2}\dfrac{a_{-n-2}}{a_{-n}}}}}. \tag{96}$$

Again the ratio a_{-n-2}/a_{-n} can be neglected for a sufficiently large value of n, and upon running up the fraction an expression is obtained for a_{l-2}/a_l in terms of b. The reciprocal of this ratio must equal the value of a_l/a_{l-2} obtained from (95). Thus (95) and (96), when properly related, constitute a transcendental equation in b whose roots are the characteristic values. When these values have been determined, the continued fractions provide an easy means of computing the coefficients a_n in terms of any arbitrary coefficient a_l.

2.5 **Termination of the Series.** In general the series expansions of the U- and V-functions extend from minus to plus infinity. However, by a proper choice of ρ they may be terminated in certain cases at $n = 0$ or $n = 1$ and extended over positive values of n only. Thus for the even series

$$\frac{a_{-2}}{a_0} = -\frac{A_0}{B_{-2} + C_{-4}\dfrac{a_{-4}}{a_{-2}}}, \tag{97}$$

with

$$A_0 = \frac{\rho(\rho - 1)}{(2\rho + 2a - 1)(2\rho + 2a + 1)}. \tag{98}$$

Then if a_0 is finite and $a \neq \pm\frac{1}{2}$, a_{-2} vanishes as $\rho \to 0$. Moreover, it follows from (96) that all the remaining coefficients a_n with negative n vanish in the same manner, while the ratio a_{n-2}/a_n remains finite. Since $a_{-2} = 0$, the coefficients with positive subscripts can all be expressed in terms of a_0. From (96) and (95) one obtains

$$\begin{aligned} \frac{a_2}{a_0} &= -\frac{B_0}{A_2}, \\ \frac{a_2}{a_0} &= -\frac{C_0}{B_2 + A_4 \dfrac{a_4}{a_2}}. \end{aligned} \tag{99}$$

As in Sec. 2.4 the separation constant b is to be determined such that the two expressions for the ratio a_2/a_0 are equal. The treatment of the odd series is identical.

Since the functions $J_p(cz)$ and $N_p(cz)$ are finite for negative orders p, it is apparent that the functions U_1, U_2, U_3, and U_6 defined in Sec. 2.2 are summed only over positive values of n, starting at $n = 0$ or $n = 1$ as l is even or odd. The remaining functions U_3, U_4, U_7, U_8 contain terms involving negative values of n in virtue of the factor $(n + \rho)!/(n + \rho + 2a)!$. Thus if $a = m$, a positive integer, the limit of $a_n(n + \rho)!/(n + \rho + 2m)!$ as $\rho \to 0$ is finite when $n > -2m - 1$, but zero for all values equal to or less than this number. The series again break off on the negative side, but at $n = -2m$ and $n = -2m + 1$ for the even and odd cases respectively. When a is a negative integer the series also terminate. If $a = -m$, the even and odd series start at $n = 2m$ and $n = 2m + 1$ respectively, where m is a positive integer, and extend to infinity in the positive direction. Finally, when a is not an integer, all terms of the summation, both positive and negative, must be included.

Similar considerations hold for the V-functions. If $a = m$, the functions V_1, V_3, V_5, V_6, V_7, V_8 are summed over positive values of n only, starting at $n = 0$ or $n = 1$. Actually V_1 and V_3 are infinite in virtue of a factor $1/\rho$, but since this factor is common to each term of the series it can be removed. The summations for the functions V_2 and V_4 extend from negative to positive infinity, *even when a is a positive integer*. The ratio a_n/ρ for negative values of n was shown to be finite.

Since $\mathfrak{T}^m_{n+\rho+2m}(z)$ for $n < 0$ and $\mathfrak{T}^m_{n+\rho}$ for $n < -2m$ contain a factor $1/\rho$, the products

$$a_n \mathfrak{T}^m_{n+\rho+2m} \quad \text{and} \quad a_n \frac{(n+\rho)!}{(n+2m+\rho)!} T^m_{n+\rho}$$

are likewise finite.

2.6 **Mathieu Functions.** The special case $a = -\frac{1}{2}$ is of particular interest since it pertains to the Mathieu functions. Taking $\rho = 0$, one now obtains

$$A_n = C_n = \tfrac{1}{4},$$

$$B_n = \frac{b - n^2}{c^2} - \frac{1}{2},$$

and the recursion formula reduces to

$$a_{n+2} + a_{n-2} - \left[2 + \frac{4}{c^2}(n^2 - b)\right] a_n = 0. \tag{100}$$

Unlike the other cases, this relation is the same for both positive and negative values of n. Moreover, the Bessel and Gegenbauer functions are now either even or odd functions of the independent variable z. We shall construct U- and V-functions that satisfy one or the other of two conditions. For the first type we shall take

$$a_n = (-1)^n a_{-n}. \tag{101}$$

The series fails to break off, but in virtue of the symmetry of positive and negative terms it can be summed from $n = 0$ or $n = 1$ to $n = \infty$. In this case a_0 differs from zero. When the index l of the separation constant is even one obtains b from the relation

$$\frac{a_2}{a_0} = 1 - \frac{2b}{c^2} = -\frac{1}{4B_2 + a_4/a_2} \tag{102}$$

which can be expanded as a continuous fraction as in (95). Likewise when l is odd, the characteristic values of b are given by the roots of

$$\frac{a_3}{a_1} = 3 + \frac{4}{c^2}(1 - b) = -\frac{1}{4B_3 + a_5/a_3}. \tag{103}$$

For the second type we shall have

$$a_n = -(-1)^n a_{-n}. \tag{104}$$

Obviously $a_0 = 0$ and the characteristic values of b corresponding to even values of l are obtained from

$$\frac{a_4}{a_2} = 2 + \frac{4}{c^2}(4 - b) = -\frac{1}{4B_4 + a_6/a_4}, \tag{105}$$

while the roots of

$$\frac{a_3}{a_1} = 1 + \frac{4}{c^2}(1 - b) = -\frac{1}{4B_3 + a_5/a_3} \tag{106}$$

are the characteristic values of b associated with odd values of l. The relation of the functions constructed in this way to the Mathieu functions defined by other authors will be shown below.

The U- and V-functions themselves have certain properties that are peculiar to the special case $a = -\frac{1}{2}$, $\rho \to 0$. We now find $(n + \rho)!/(n + 2a + \rho)! = n$, $n + \rho + a + \frac{1}{2} = n$, and since for any Bessel function of integral order $Z_n = (-1)^n Z_{-n}$, there are two cases to be considered.

1. *Even Functions* $a_n = (-1)^n a_{-n}$. Then U_3, U_4, U_7, U_8 vanish, for the positive half of the series is equal but opposite in sign to the negative. The remaining functions U_1, U_2, U_5, U_6 are of the form

$$\sum_n{}' a_n Z_{\pm n}(cz).$$

Likewise V_7 and V_8 vanish, while the remaining V-functions can be put into the form

$$\sum_n{}' a_n i^n \cosh n\psi,$$

where $\psi = \cosh^{-1} z$, and are therefore proportional to each other. Thus when $\rho = 0$ we have only one independent V-function. A second solution in this case must be found by a limiting process. Mathieu functions constructed in terms of $\cosh n\psi$ alone are even functions of ψ, and it is apparent that this has nothing to do with the odd or even character of l.

2. *Odd Functions*, $a_n = -(-1)^n a_{-n}$. The functions U_1, U_2, U_5, U_6 now vanish while U_3, U_4, U_7, U_8 are all of the form

$$\frac{\sqrt{z^2 - 1}}{cz} \sum_n{}' n a_n Z_{\pm n}.$$

Likewise V_5 and V_6 vanish and the remaining V-functions are all of the form

$$\sum_n{}' i^n a_n \sinh n\psi$$

and hence proportional to each other. They are odd functions of ψ, for even as well as odd values of l.

2.7 **Convergence.** The recursion formulas associated with equations of Mathieu type are of three terms. Consequently the convergence theory is more difficult than that of hypergeometric equations whose recursion formulas are always of two terms. In the present section we shall indicate the nature of the convergence and the conditions which appear to justify certain operations carried out in the remainder of this paper. The correctness of the results is supported also by extensive numerical calculations. A more thorough investigation is nevertheless desirable.

According to (93) the recursion formula relating the expansion coefficients can be written in the form

$$\frac{a_n}{a_{n-2}} = -\frac{C_{n-2}}{B_n + A_{n+2}\dfrac{a_{n+2}}{a_n}} \tag{107}$$

Let $r_n = a_n/a_{n-2}$. Then as $n \to \infty$,

$$r_n \to \frac{c^2}{2c^2 - 4b + 4n^2 - c^2 r_{n+2}}. \tag{108}$$

There are two possibilities. The separation constant b may be such that the denominator of (107) approaches zero with increasing n. The ratio r_n increases as n^2 and the series $\sum a_n$ diverges for all values of c. But b may also be chosen such that $r_{n+2} \to 0$. This was the procedure described in the foregoing sections and it is now apparent that for these characteristic values of b

$$r_n \to \frac{c^2}{4n^2}. \tag{109}$$

The series of coefficients a_n converges accordingly.

It is interesting to note the origin of these two sets of b, the one leading to divergent, the other to convergent series. The recursion formulas for the coefficients constitute a system of *difference* equations. A two-term difference equation is the counterpart of a first order dif-

ferential equation. One is free to specify the behavior of the solution in the neighborhood of $n = 0$, but the character of the solution as $n \to \infty$ is then fixed. A three-term difference equation corresponds on the other hand to a differential equation of second order, to which there are two independent solutions. Thus we may specify the value of the ratio r_n for small values of n, but this condition is no longer sufficient to determine the behavior of r_n at infinity. From the theory of difference equations it is easy to find asymptotic solutions when $n \to \infty$. The reader may verify that there are in fact two, the one becoming infinite, the other vanishing with increasing n. Of these two we choose in effect the latter, continue it analytically into the region of small n, and then select b such that the function r_n assumes the proper value at $n = 0$ or 1.

Consider now the U- and V-functions and examine the behavior of Bessel and Gegenbauer functions whose order is very much larger than the argument. Let $cz = p \operatorname{sech} \alpha \leqq p$. As $p \to \infty$, one must have $\alpha \to \infty$ if cz is to remain finite, or $\tanh \alpha \to 1$. Then according to Watson[8], when α is any fixed positive number and p is large and positive, one obtains the following asymptotic representation:

$$J_p(p \operatorname{sech} \alpha) \to \frac{e^{p(\tanh \alpha - \alpha)}}{\sqrt{2\pi p \tanh \alpha}}. \tag{110}$$

Since $\alpha \gg \tanh \alpha$, (110) vanishes exponentially with increasing p. On the other hand

$$N_p(p \operatorname{sech} \alpha) \to -\frac{e^{p(\alpha - \tanh \alpha)}}{\sqrt{\tfrac{1}{2}\pi p \tanh \alpha}}. \tag{111}$$

This function grows large without limit as $p \to \infty$, and the same is true of $J_{-p}(z)$ when p is not an integer. Nonetheless the corresponding U-functions converge in virtue of the rapid decrease in the coefficients a_n. Thus if $p = n + \rho + a + \tfrac{1}{2}$,

$$\frac{a_n N_p(cz)}{a_{n-2} N_{p-2}(cz)} \to \frac{c^2}{4n^2} e^{2(\alpha - \tanh \alpha)} \tag{112}$$

as $n \to \infty$. The asymptotic formulas (110) and (111) hold only when $p \gg cz \gg 1$. Hence the convergence of the U-functions has been demonstrated only for large values of the argument. Presumably the expansions (73)–(80) are convergent throughout the entire domain ex-

[8] G. N. Watson, "Theory of Bessel Functions," p. 243, Cambridge University Press, 1922.

terior to a circle of unit radius about the origin $z = 0$. From a practical point of view, however, these series are useful in most cases only when z is large. The functions $N_p(cz)$ grow large when the order p is equal or greater than the argument. If $z \simeq 1$, the first few terms increase rapidly and many terms must be employed before the rate of decrease in the coefficients a_n is sufficient to establish convergence.

When $|z| < 1$, all series containing terms in $N_p(cz)$ or $J_{-p}(cz)$ diverge. This is easily shown when $cz \ll 1$. For then

$$\frac{a_n N_p(cz)}{a_{n-2} N_{p-2}(cz)} \simeq \frac{1}{z^2}. \tag{113}$$

Exceptions occur when $a = m$ or $a = -\frac{1}{2}$. It has been shown that in these cases certain series can be terminated. All expansions in terms of Bessel functions of the first kind summed over positive orders only converge over the entire z-plane, including the points $z = \pm 1$.

The convergence of the V-functions is determined by the behavior of Gegenbauer functions of larger order. The required asymptotic representations are given by Hobson.[9] If the coefficients a_n diminish according to (109), it can be shown that the expansions (85–92) are convergent both in the interior of a circle of unit radius about the origin and in the region exterior to that circle. At the points $z = \pm 1$ the value of the functions may become infinite owing to a logarithmic factor or a factor of the form $(z^2 - 1)^{-\alpha}$.

2.8 **Expansions of U- and V-Functions in Power Series.** There remains the fundamental problem of establishing analytic connections between the 16 functions defined in the preceding sections. This will be accomplished by first representing the various functions in terms of power series; coefficients can then be compared. In the domain $|z| > 1$ we shall employ power series representations of Bessel and Gegenbauer functions that are absolutely convergent. Terms may be grouped and the order changed arbitrarily. From the condition that the coefficients a_n form a convergent series it can be shown that the power series expansions of U- and V-functions obtained in this way likewise converge. This in no way contradicts the fact that in general only asymptotic representations can be found in the neighborhood of an essential singularity. For these power series expansions are actually Laurent series valid within an annular region bounded internally by the circle $|z| = 1$ and

[9] Hobson, loc. cit., pp. 302–313.

externally by a circle of very large radius excluding the point at infinity.

Within this annular region we find for the first four U-functions:

$$U_1 = 2^{-a-\frac{1}{2}}\left(\frac{cz}{2}\right)^{\rho}\sum_n{}' a_n\left(\frac{cz}{2}\right)^n \times \sum_{k=0}^{\infty}\frac{1}{k!(n+\rho+a+k+\frac{1}{2})!}\left(\frac{icz}{2}\right)^{2k} \tag{114}$$

$$U_2 = 2^{-a-\frac{1}{2}}\left(\frac{cz}{2}\right)^{-(2a+\rho+1)}\sum_n{}' a_n\left(\frac{cz}{2}\right)^{-n} \times \sum_{k=0}^{\infty}\frac{1}{k!(-n-\rho-a+k-\frac{1}{2})!}\left(\frac{icz}{2}\right)^{2k} \tag{115}$$

$$U_3 = (z^2-1)^{-a}2^{a-\frac{1}{2}}\left(\frac{cz}{2}\right)^{2a+\rho}\sum_n{}' a_n\frac{(n+\rho)!}{(n+\rho+2a)!}\left(\frac{cz}{2}\right)^n \times \sum_{k=0}^{\infty}\frac{1}{k!(n+\rho+a+k+\frac{1}{2})!}\left(\frac{icz}{2}\right)^{2k} \tag{116}$$

$$U_4 = (z^2-1)^{-a}2^{a-\frac{1}{2}}\left(\frac{cz}{2}\right)^{-\rho-1}\sum_n{}' a_n\frac{(n+\rho)!}{(n+\rho+2a)!}\left(\frac{cz}{2}\right)^{-n} \times \sum_{k=0}^{\infty}\frac{1}{k!(-n-\rho-a+k-\frac{1}{2})!}\left(\frac{icz}{2}\right)^{2k} \tag{117}$$

When $|z| > 1$ the $\mathfrak{T}$-functions are represented by the series (39). In this domain the first four V-functions assume the form

$$V_1 = (-\tfrac{1}{2})^{-a}\sqrt{\pi}\,(2z)^{\rho}\sum_n{}' a_n\frac{(-n-\rho-2a-1)!}{(-n-\rho-1)!}(i2z)^n \times \sum_{r=0}^{\infty}\frac{(2r-n-\rho-1)!}{r!(r-a-n-\rho-\frac{1}{2})!}(2z)^{-2r} \tag{118}$$

$$V_2 = (-\tfrac{1}{2})^{-a}\sqrt{\pi}\,(2z)^{-(2a+\rho+1)}\sum_n{}' a_n\frac{(n+\rho)!}{(n+\rho+2a)!}(-i2z)^{-n} \times \sum_{r=0}^{\infty}\frac{(2r+n+\rho+2a)!}{r!(r+n+\rho+a+\frac{1}{2})!}(2z)^{-2r} \tag{119}$$

$$V_3 = (-\tfrac{1}{2})^{a}\sqrt{\pi}\,(z^2-1)^{-a}(2z)^{2a+\rho}\frac{\sin(2a+\rho)\pi}{\sin\rho\pi}\sum_n{}' a_n(i2z)^n \times \sum_{r=0}^{\infty}\frac{(2r-n-\rho-2a-1)!}{r!(r-n-\rho-a-\frac{1}{2})!}(2z)^{-2r} \tag{120}$$

$$V_4 = (-\tfrac{1}{2})^s \sqrt{\pi}\,(z^2-1)^{-s}(2z)^{-\rho-1} \sum_n{}' a_n(-i2z)^{-n} \times \sum_{r=0}^{\infty} \frac{(2r+n+\rho)!}{r!(r+n+\rho+a+\tfrac{1}{2})!}(2z)^{-2r} \tag{121}$$

Let s be an integer which differs from n always by an even integer. The eight solutions can be arranged as follows:

$$U_1 = 2^{-s-1}\left(\frac{cz}{2}\right)^{\rho} \sum_s{}' \left(\frac{cz}{2}\right)^{s} \times \sum_{n \gtrless s}{}' \frac{a_n i^{s-n}}{\left(\frac{s-n}{2}\right)!\left(\frac{s+n+1}{2}+\rho+a\right)!} \tag{122}$$

$$U_2 = 2^{-s-1}\left(\frac{cz}{2}\right)^{-2a-1-\rho} \sum_s{}' \left(\frac{cz}{2}\right)^{-s} \times \sum_{n \gtrless s}{}' \frac{a_n i^{n-s}}{\left(\frac{n-s}{2}\right)!\left(-\frac{s+n+1}{2}-\rho-a\right)!} \tag{123}$$

$$U_3 = 2^{s-1}(z^2-1)^{-s}\left(\frac{cz}{2}\right)^{2a+\rho} \sum_s{}' \left(\frac{cz}{2}\right)^{s} \times \sum_{n \gtrless s}{}' \frac{a_n i^{s-n}(n+\rho)!}{\left(\frac{s-n}{2}\right)!\left(\frac{s+n+1}{2}+a+\rho\right)!\,(n+\rho+2a)!} \tag{124}$$

$$U_4 = 2^{s-1}(z^2-1)^{-s}\left(\frac{cz}{2}\right)^{-\rho-1} \sum_s{}' \left(\frac{cz}{2}\right)^{-s} \times \sum_{n \gtrless s}{}' \frac{a_n i^{n-s}(n+\rho)!}{\left(\frac{n-s}{2}\right)!\left(-\frac{s+n+1}{2}-a-\rho\right)!\,(n+\rho+2a)!} \tag{125}$$

$$V_1 = (-\tfrac{1}{2})^{-s}\sqrt{\pi}\,(2z)^{\rho} \sum_s{}' (2z)^{s}(-s-\rho-1)! \times \sum_{n \gtrless s}{}' \frac{a_n i^{n}(-n-\rho-2a-1)!}{\left(\frac{n-s}{2}\right)!\left(-\frac{s+n+1}{2}-a-\rho\right)!\,(-n-\rho-1)!} \tag{126}$$

$$V_2 = (-\tfrac{1}{2})^{-s}\sqrt{\pi}\,(2z)^{-2a-1-\rho} \sum_s{}' (2z)^{-s}(s+2a+\rho)! \times \sum_{n \gtrless s}{}' \frac{a_n i^{n}(n+\rho)!}{\left(\frac{s-n}{2}\right)!\left(\frac{s+n+1}{2}+a+\rho\right)!\,(n+2a+\rho)!} \tag{127}$$

$$V_3 = (-\tfrac{1}{2})^s\sqrt{\pi}\,(z^2-1)^{-s}(2z)^{2a+\rho}\,\frac{\sin(2a+\rho)\pi}{\sin\rho\pi}$$
$$\times \sum_s{}' (2z)^s(-s-2a-\rho-1)!$$
$$\times \sum_{n\geq s}{}' \frac{a_n i^n}{\left(\frac{n-s}{2}\right)!\left(-\frac{s+n+1}{2}-a-\rho\right)!} \tag{128}$$

$$V_4 = (-\tfrac{1}{2})^s\sqrt{\pi}\,(z^2-1)^{-s}(2z)^{-\rho-1}\sum_s{}' (2z)^{-s}(s+\rho)!$$
$$\times \sum_{n\geq s}{}' \frac{a_n i^n}{\left(\frac{s-n}{2}\right)!\left(\frac{s+n+1}{2}+a+\rho\right)!} \tag{129}$$

2.9 **Relations between the Functions.** Since the eight functions defined in Sec. 2.8 satisfy one and the same equation, there can be at most two that are linearly independent within a common region of convergence. One observes that, so far as powers of z are concerned, each U-function is identical with the V-function of the same subscript, and we conclude that the two are proportional. Thus

$$U_1 = K_1V_1\,, \qquad U_2 = K_2V_2\,,$$
$$U_3 = K_3V_3\,, \qquad U_4 = K_4V_4\,. \tag{130}$$

The proportionality constant K is called the joining factor. Upon equating the power series expansions of U and V the factor K is given as the ratio of coefficients of corresponding powers of z.

To obtain further relations between the various solutions we make use of the formula defining the Bessel function of the second kind,

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} \tag{131}$$

together with the functional relations between the Gegenbauer functions given in Sec. 1.3. Through these it is easy to establish the following connections.

Relations among the U-functions

$$U_1 = \frac{U_2\sin(a+\rho)\pi + (-1)^n U_4}{\cos(a+\rho)\pi} \tag{132}$$

$$U_2 = -\frac{U_4\sin(a+\rho)\pi + (-1)^n U_3}{\cos(a+\rho)\pi} \tag{133}$$

$$U_3 = \frac{U_7 \sin (a+\rho)\pi + (-1)^n U_8}{\cos (a+\rho)\pi} \tag{134}$$

$$U_4 = -\frac{U_8 \sin (a+\rho)\pi + (-1)^n U_7}{\cos (a+\rho)\pi} \tag{135}$$

$$U_5 = -\frac{U_1 \sin (a+\rho)\pi + (-1)^n U_2}{\cos (a+\rho)\pi} \tag{136}$$

$$U_6 = \frac{U_2 \sin (a+\rho)\pi + (-1)^n U_1}{\cos (a+\rho)\pi} \tag{137}$$

$$U_7 = -\frac{U_3 \sin (a+\rho)\pi + (-1)^n U_4}{\cos (a+\rho)\pi} \tag{138}$$

$$U_8 = \frac{U_4 \sin (a+\rho)\pi + (-1)^n U_3}{\cos (a+\rho)\pi} \tag{139}$$

Relations among the V-functions

$$V_1 = (-1)^{-2s} \frac{\sin \rho\pi}{\sin (2a+\rho)\pi} V_3 \tag{140}$$

$$V_2 = (-1)^{-2s} V_4 \tag{141}$$

$$V_5 = V_6 \tag{142}$$

$$V_7 = V_8 \tag{143}$$

$$(-1)^s \frac{2}{\pi} \sin a\pi V_1 = \frac{\sin \rho\pi}{\sin (2a+\rho)\pi} V_7 - V_5 \tag{144}$$

$$(-1)^s \frac{2}{\pi} \sin a\pi V_2 = V_7 - V_5 \tag{145}$$

$$(-1)^s \pi \cos (a+\rho)\pi V_5 = \sin \rho\pi (V_4 - V_3) \tag{146}$$

$$(-1)^s \pi \cos (a+\rho)\pi V_7 = \sin (2a+\rho)\pi (V_2 - V_1) \tag{147}$$

Relations between U- and V-functions

The power series expansions of the functions U_1, U_2, U_3, U_4, V_1, V_2, V_3, V_4 converge within the domain $1 < |z| < \infty$ when b has a characteristic value ensuring the absolute convergence of the coefficients a_n. Since U- and V-functions of equal subscript have power series of identical form, they must be proportional to each other and the proportionality factor $K_j = U_j/V_j$, $(j = 1, 2, 3, 4)$, must equal the ratio of the coefficients of z^s in the corresponding series. Thus there are an infinite number of expressions for the joining factors, for s is any positive or negative integer differing from n or l by an even integer.

$$K_1 = \frac{(-1)^{a+s/2} c^{s+\rho}}{\sqrt{2\pi}\, 2^{2(s+a+\rho)}(-s-\rho-1)!} \times \frac{\sum'_{n\geq s} \frac{i^{-n} a_n}{\left(\frac{s-n}{2}\right)!\left(\frac{s+n+1}{2}+a+\rho\right)!}}{\sum'_{n\geq s} \frac{i^n(-n-2a-1-\rho)!\, a_n}{\left(\frac{n-s}{2}\right)!\left(-\frac{s+n+1}{2}-a-\rho\right)!(-n-1-\rho)!}} \tag{148}$$

$$K_2 = \frac{(-1)^{a-s/2} 2^{2(s+a+1+\rho)}}{\sqrt{2\pi}\, c^{s+2a+1+\rho}(s+2a+\rho)!} \times \frac{\sum'_{n\geq s} \frac{i^n a_n}{\left(\frac{n-s}{2}\right)!\left(-\frac{s+n+1}{2}-a-\rho\right)!}}{\sum'_{n\geq s} \frac{i^n(n+\rho)!\, a_n}{\left(\frac{s-n}{2}\right)!\left(\frac{s+n+1}{2}+a+\rho\right)!(n+2a+\rho)!}} \tag{149}$$

$$K_3 = \frac{(-1)^{-a+s/2} c^{s+2a+\rho}}{\sqrt{2\pi}\, 2^{2(s+a+\rho)}(-s-2a-1-\rho)!} \times \frac{\sum'_{n\geq s} \frac{i^{-n}(-n-2a-1-\rho)!\, a_n}{\left(\frac{s-n}{2}\right)!\left(\frac{s+n+1}{2}+a+\rho\right)!(-n-1-\rho)!}}{\sum'_{n\geq s} \frac{i^n a_n}{\left(\frac{n-s}{2}\right)!\left(-\frac{s+n+1}{2}-a-\rho\right)!}} \tag{150}$$

$$K_4 = \frac{(-1)^{-a-s/2} 2^{2(s+a+1+\rho)}}{\sqrt{2\pi}\, c^{s+1+\rho}(s+\rho)!} \times \frac{\sum'_{n\geq s} \frac{i^n(n+\rho)!\, a_n}{\left(\frac{n-s}{2}\right)!\left(-\frac{s+n+1}{2}-a-\rho\right)!(n+2a+\rho)!}}{\sum'_{n\geq s} \frac{i^n a_n}{\left(\frac{s-n}{2}\right)!\left(\frac{s+n+1}{2}+a+\rho\right)!}}. \tag{151}$$

Every value of s must, of course, lead to one and the same value of K_f.

2.10 **Summary of Relations when $a = m$.** When $a = m$, a positive integer, and $\rho \to 0$, the 16 U- and V-functions fall into two groups of independent solutions. The following table is so arranged that all functions in a group or column are identical.

TABLE II

Group 1	*Group 2*
$-\rho V_1$	V_2
$-\rho V_3$	V_4
V_5	U_2/K_2
V_6	U_4/K_4
V_7	$(-1)^{n+m+1}U_5/K_2$
V_8	$(-1)^{n+m+1}U_7/K_4$
$-\rho U_1/K_1$	
$-\rho U_3/K_3$	
$(-1)^{n+m+1}\rho U_5/K_1$	
$(-1)^{n+m+1}\rho U_7/K_3$	

By proper choice of s the joining factors reduce to the following. For l and n even:

$$K_1 = -\rho \frac{2^{-m-1}a_0}{(m+\frac{1}{2})!}[V_3(0)]^{-1} \tag{152}$$

$$K_2 = -\frac{2^{-m+1}}{c(m-\frac{3}{2})!\,(\rho-2m)!\,a_{-2m}} V_3(0) \tag{153}$$

$$K_3 = \rho\frac{2^{m-1}(m-\frac{3}{2})!\,(\rho-2m)!\,a_{-2m}}{\pi}[V_3(0)]^{-1} \tag{154}$$

$$K_4 = \frac{2^{m+1}(m+\frac{1}{2})!}{\pi c a_0} V_3(0) \tag{155}$$

where

$$V_3(0) = V_3(a, c; z)\,|_{s=0} = \frac{1}{2^m\sqrt{\pi}} \sum_{n=0}^{\infty}{}' a_n \frac{\left(\frac{n-1}{2}\right)!}{\left(\frac{n}{2}+m\right)!}. \tag{156}$$

For l and n odd:

$$K_1 = -\rho \frac{2^{-m-1}ca_1}{(m+\frac{3}{2})!}[V_3'(0)]^{-1} \tag{157}$$

$$K_2 = \frac{2^{-m+l}}{c^2(m-\frac{s}{2})!\,(\rho+1-2m)!\,a_{1-2m}} V_s'(0) \tag{158}$$

$$K_3 = -\rho \frac{2^{m-l} c(m-\frac{s}{2})!\,(\rho+1-2m)!\,a_{1-2m}}{\pi} [V_s'(0)]^{-1} \tag{159}$$

$$K_4 = \frac{2^{m+l}(m+\frac{s}{2})!}{\pi c^2 a_l} V_s'(0), \tag{160}$$

where

$$V_s'(0) = \frac{d}{dz} V_s(a, c; z)\Big|_{z=0} = \frac{i}{2^{m-l}\sqrt{\pi}} \sum_{n=1}{}' a_n \frac{\left(\frac{n}{2}\right)!}{\left(\frac{n-1}{2}+m\right)!} \tag{161}$$

The coefficients a_{-2m} and a_{1-2m} vanish as $\rho \to 0$ and $m > 0$, but the products $(\rho - 2m)!a_{-2m}$ and $(\rho + 1 - 2m)!a_{1-2m}$ are finite.

For both even and odd values of n

$$K_1K_4 = K_2K_3 = -\frac{2\rho}{\pi c}.$$

2.11 **Summary of Relations when $a = -\frac{1}{2}$.** When $a = -\frac{1}{2}$ and $\rho = 0$ the joining factors are reduced to the following by proper choice of s. These formulas apply both to the even and the odd Mathieu functions.

For l and n even:

$$K_1 = -i\sqrt{\frac{2}{\pi}} a_0 \left[a_0 + 2\sum_{n=2}{}' a_n\right]^{-1} \tag{163}$$

$$K_2 = -i\sqrt{\frac{2}{\pi}} a_0 \left[a_0 + 2\sum_{n=2}{}' a_{-n}\right]^{-1} \tag{164}$$

$$K_3 = i\frac{c}{8}\sqrt{\frac{2}{\pi}} (a_2 - a_{-2}) \left[\sum_{n=2} n a_n\right]^{-1} \tag{165}$$

$$K_4 = -i\frac{c}{8}\sqrt{\frac{2}{\pi}} (a_2 - a_{-2}) \left[\sum_{n=2} n a_{-n}\right]^{-1} \tag{166}$$

For l and n odd:

$$K_1 = \frac{c}{2\sqrt{2\pi}} (a_{-1} - a_1) \left[\sum_{n=1} n a_n\right]^{-1} \tag{167}$$

$$K_2 = \frac{c}{2\sqrt{2\pi}}(a_{-1} - a_1)\left[\sum_{n=1}^{\infty} n a_{-n}\right]^{-1} \tag{168}$$

$$K_3 = -\frac{1}{2\sqrt{2\pi}}(a_{-1} + a_1)\left[\sum_{n=1}^{\infty} a_n\right]^{-1} \tag{169}$$

$$K_4 = \frac{1}{2\sqrt{2\pi}}(a_{-1} + a_1)\left[\sum_{n=1}^{\infty} a_{-n}\right]^{-1} \tag{170}$$

Even Functions, $a_n = (-1)^n a_{-n}$. The functions U_3, U_4, U_7, U_8, V_7, V_8 vanish as $\rho \to 0$, as do also the joining factors K_3 and K_4, and

$$K_1 = (-1)^n K_2. \tag{171}$$

The remaining functions again fall into two independent groups.

TABLE III

Group 1	*Group* 2
V_1	U_5
V_2	$(-1)^n U_6$
V_3	
$-V_4$	
$i\frac{\pi}{2} V_5$	
$i\frac{\pi}{2} V_6$	
U_1/K_1	
U_2/K_2	

As before, all functions occurring in one group or column are identical and represent one independent solution. The Gegenbauer functions for $a = -\frac{1}{2}$ can be expressed in terms of exponential and hyperbolic functions; by writing $z = \cosh \psi$ and doubling up the series one obtains:

l and n even,

$$V_1 = i\sqrt{\frac{\pi}{2}}\left[a_0 + 2\sum_{n=2}^{\infty}{}' a_n i^n \cosh n\psi\right] \tag{172}$$

$$U_1 = a_0 J_0(cz) + 2\sum_{n=2}^{\infty}{}' a_n J_n(cz) \tag{173}$$

$$U_5 = a_0 N_0(cz) + 2\sum_{n=2}^{\infty}{}' a_n N_n(cz) \tag{174}$$

l and n odd,

$$V_1 = i\sqrt{2\pi} \sum_{n=1}^{\infty}{}' a_n i^n \cosh n\psi \tag{175}$$

$$U_1 = 2 \sum_{n=1}^{\infty}{}' a_n J_n(cz) \tag{176}$$

$$U_3 = 2 \sum_{n=1}^{\infty}{}' a_n N_n(cz). \tag{177}$$

V_1 and U_1 are Mathieu functions of the first kind and these representations converge over the entire z-plane. U_3 is a Mathieu function of the second kind, but the convergence of (174) and (177) is restricted to the region $|z| > 1$. A second solution of Mathieu's equation valid in the neighborhood of $z = 1$ has been discussed by Ince[10], but he fails to give the analytic connection with a corresponding solution valid when $z \to \infty$. In the next section we shall obtain another form of the second solution and the desired connections through application of a limiting process to the vanishing functions V_7 and V_8.

Odd Functions, $a_n = -(-1)^n a_{-n}$. In this case U_1, U_2, U_5, U_6, V_5, V_6, and the joining factors K_1 and K_2 vanish as $\rho \to 0$. Also

$$K_3 = (-1)^n K_4. \tag{178}$$

The remaining functions fall into two independent groups as in Table IV, the functions in any one column being all identical.

TABLE IV

Group 1	*Group* 2
V_1	U_7
$-V_2$	$(-1)^n U_8$
V_3	
V_4	
$i\frac{\pi}{2} V_7$	
$i\frac{\pi}{2} V_8$	
U_3/K_3	
U_4/K_4	

[10] E. L. Ince, Proc. Edinburgh Math. Soc., pp. 2-13, 1914-1915.

Upon doubling up the series so that the summation extends only over positive values of n one obtains

$$V_3 = i\sqrt{2\pi} \sum_{n=1,2}^{\infty}{}' a_n i^n \sinh n\psi, \tag{179}$$

$$U_3 = 2\frac{\sqrt{z^2-1}}{cz} \sum_{n=1,2}^{\infty}{}' a_n n J_n(cz), \tag{180}$$

$$U_7 = 2\frac{\sqrt{z^2-1}}{cz} \sum_{n=1,2}^{\infty}{}' a_n n N_n(cz). \tag{181}$$

When l is even these series start from $n = 2$, and from $n = 1$ when l is odd. As in the case of the even functions a second solution valid when $|z| \sim 1$ is missing, but will now be found by a limiting operation applied to the vanishing functions V_3 and V_8.

2.12 **Second Solutions of the Mathieu Equation.** The object of this section is to find a second solution of Mathieu's equation in the domain $|z| \sim 1$ and to join it analytically with representations of the same function valid when $|z| \gg 1$. In Sec. 2.11 it was shown that V_7 and V_8 for the even functions, and V_5 and V_6 for the odd functions, vanish as $\rho \to 0$. The zero is of the first order in ρ and hence can be removed if one divides by ρ before passing to the limit. Since ρ is an arbitrary constant, the functions obtained in this manner are still solutions of the differential equation.

Even Functions, $a_n = (-1)^n a_{-n}$. Formally

$$V_7 = V_8 = \sqrt{\frac{2}{\pi}} \sum_{n}{}' a_n i^n \sinh (n+\rho)\psi, \tag{182}$$

where the a_n are also functions of ρ. They vanish as $\rho \to 0$, since $a_n i^n \sinh n\psi = -a_{-n} i^{-n} \sinh (-n\psi)$. We now divide V_7 and V_8 by ρ, differentiate numerator and denominator with respect to ρ and pass to the limit $\rho = 0$.

$$\lim_{\rho\to 0} \frac{V_7}{\rho} = \lim_{\rho\to 0} \frac{V_8}{\rho} = \sqrt{\frac{2}{\pi}} \sum_{n}{}' i^n [a_n' \sinh n\psi + a_n \psi \cosh n\psi], \tag{183}$$

where

$$a_n' = \frac{d}{d\rho} a_n \Big|_{\rho=0}. \tag{184}$$

In virtue of the relation

$$a_n' = -(-1)^n a_{-n}', \tag{185}$$

which will be established in the following section, the function defined by (183) is finite and does not vanish. It differs from all other V-functions in that it is an odd function of ψ, and it must, therefore, be a second solution of the Mathieu equation. It is this solution that was found by Ince[10] in another manner.

To find a relation between U- and V-functions of the second kind we make use of (136) for the case $a = -\frac{1}{2}$.

$$U_3 = \frac{U_1 \cos \rho\pi - (-1)^n U_2}{\sin \rho\pi}. \tag{186}$$

Upon replacing U_1 and U_2 by their equivalents in terms of V-functions and allowing $\rho \to 0$, one obtains

$$U_3 = \frac{1}{\pi} \lim_{\rho \to 0} \frac{d}{d\rho} [K_1 V_1 - (-1)^n K_2 V_2]. \tag{187}$$

Let

$$K' = \left.\frac{dK}{d\rho}\right|_{\rho=0}, \qquad V' = \left.\frac{dV}{d\rho}\right|_{\rho=0}. \tag{188}$$

Since $K_1 = (-1)^n K_2$, $V_1 = V_2$, we have

$$U_3 = \frac{1}{\pi} \{[K_1' - (-1)^n K_2'] V_1 + K_1 [V_1' - V_2']\}. \tag{189}$$

From (85), (86), and (55), when $\rho \neq 0$,

$$V_1 = i \sqrt{\frac{\pi}{2}} \sum_n{}' a_n i^n e^{(n+\rho)\psi} \tag{190}$$

$$V_2 = i \sqrt{\frac{\pi}{2}} \sum_n{}' a_n i^n e^{-(n+\rho)\psi}. \tag{191}$$

Thus

$$\begin{aligned} V_1' - V_2' &= i\sqrt{2\pi} \sum_n{}' i^n [a_n' \sinh n\psi + a_n \psi \cosh n\psi] \\ &= i\pi \frac{V_7}{\rho} = i\pi \frac{V_8}{\rho}. \end{aligned} \tag{192}$$

It appears that the solution U_3 valid when z is large is in fact a linear combination of solutions of the first and second kinds established for the region $|z| \sim 1$.

We proceed now to calculate the derivatives of the joining factors K. In (148) and (149) put $a = -\frac{1}{2}$ and obtain

$$K_1 = \frac{i\sqrt{\frac{2}{\pi}}\left(\frac{c}{4}\right)^{s+\rho}\sum_{n\geq s}'\frac{a_n i^{n-s}}{\left(\frac{s-n}{2}\right)!\left(\frac{s+n}{2}+\rho\right)!}}{(-s-1-\rho)!\sum_{n\geq s}'\frac{a_n i^n(n+\rho)}{\left(\frac{n-s}{2}\right)!\left(-\frac{s+n}{2}-\rho\right)!}} \tag{193}$$

$$K_2 = -\frac{i\sqrt{\frac{2}{\pi}}\left(\frac{c}{4}\right)^{-s-\rho}\sum_{n\geq s}'\frac{a_n i^{n-s}}{\left(\frac{n-s}{2}\right)!\left(-\frac{s+n}{2}-\rho\right)!}}{(s-1+\rho)!\sum_{n\geq s}'\frac{a_n i^n(n+\rho)}{\left(\frac{s-n}{2}\right)!\left(\frac{s+n}{2}+\rho\right)!}}. \tag{194}$$

For even values of n let $s = 0$. After simplification we obtain

$$K_1' = -K_2'$$

$$= K_1 \ln\frac{c}{4} - i\sqrt{\frac{2}{\pi}}\left\{\frac{a_0 C - 2\sum_{n=2}^{\infty}{}'\frac{a_n}{n}}{a_0 + 2\sum_{n=2}^{\infty}{}' a_n} - \frac{2a_0\sum_{n=2}^{\infty}{}'\left[a_n' + a_n\left\{\sigma\left(\frac{n}{2}-1\right)+\frac{1}{n}\right\}\right]}{\left[a_0 + 2\sum_{n=2}^{\infty}{}' a_n\right]^2}\right\} \tag{195}$$

where

$$C = 0.577216, \qquad \sigma(x) = \sum_{n=1}^{x}\frac{1}{n}.$$

For odd values of n let $s = 1$ in K_1 and $s = -1$ in K_2. Then

$$K_1' = K_2' = K_1 \ln\frac{c}{4} + \frac{c}{\sqrt{2\pi}}\left\{\frac{-a_1(C+\frac{1}{2}) + 2\sum_{n=3}^{\infty}{}'\frac{a_n}{n^2-1}}{\sum_{n=1}^{\infty}{}' a_n n} + a_1\sum_{n=1}^{\infty}{}'\frac{\left[a_n' n + a_n\left\{1 + n\sigma\left(\frac{n-1}{2}\right)\right\}\right]}{\left[\sum_{n=1}^{\infty}{}' a_n n\right]^2}\right\}. \tag{196}$$

In general

$$K_1' = -(-1)^n K_2', \tag{197}$$

$$U_5 = \frac{2}{\pi} K_1' V_1 + iK_1 \frac{V_2}{\rho}. \tag{198}$$

Odd Functions, $a_n = -(-1)^n a_{-n}$. Formally

$$V_5 = V_6 = \sqrt{\frac{2}{\pi}} \sum_n' a_n i^n \cosh (n + \rho)\psi. \tag{199}$$

They vanish as $\rho \to 0$, since $a_n i^n \cosh n\psi = -a_{-n} i^{-n} \cosh (-n\psi)$. Upon division by ρ and passing to the limit $\rho \to 0$ one obtains

$$\lim_{\rho \to 0} \frac{V_5}{\rho} = \lim_{\rho \to 0} \frac{V_6}{\rho} = \sqrt{\frac{2}{\pi}} \sum_n' i^n [a_n' \cosh n\psi + a_n \psi \sinh n\psi]. \tag{200}$$

This is an even function of ψ, for

$$a_n' = (-1)^n a_{-n}', \tag{201}$$

as will be shown below.

To establish a relation between U- and V-functions use is made of (138).

$$U_7 = \frac{U_3 \cos \rho\pi - (-1)^n U_4}{\sin \rho\pi}. \tag{202}$$

Replacing U_3 and U_4 by their equivalent V-functions and letting $\rho \to 0$, we have

$$U_7 = \frac{1}{\pi} \lim_{\rho \to 0} \frac{d}{d\rho} [K_3 V_3 - (-1)^n K_4 V_4], \tag{203}$$

or

$$U_7 = \frac{1}{\pi} \{[K_3' - (-1)^n K_4'] V_3 + K_3 [V_3' - V_4']\}. \tag{204}$$

For arbitrary values of ρ

$$V_3 = i \sqrt{\frac{\pi}{2}} \sum_n' a_n i^n e^{(n+\rho)\psi}, \tag{205}$$

$$V_4 = -i \sqrt{\frac{\pi}{2}} \sum_n' a_n i^n e^{-(n+\rho)\psi}. \tag{206}$$

Thus

$$V_3' - V_4' = i\sqrt{2\pi} \sum_n{}' i^n[a_n' \cosh n\psi + a_n\psi \sinh n\psi]$$

$$= i\pi \lim_{\rho\to 0} \frac{V_5}{\rho} = i\pi \lim_{\rho\to 0} \frac{V_6}{\rho}. \tag{207}$$

As before we find that U_7, valid when z is large, is a linear combination of two independent solutions which have been established as useful in the neighborhood of $z = 1$.

The formulas for K_3 and K_4 reduce to the following when $a = -\frac{1}{2}$.

$$K_3 = -\frac{i\left(\frac{c}{4}\right)^{s+\rho-1} \sum_{n\geq s}' \frac{a_n i^{s-n}(n+\rho)}{\left(\frac{s-n}{2}\right)!\left(\frac{s+n}{2}+\rho\right)!}}{2\sqrt{2\pi}\,(-s-\rho)! \sum_{n\geq s}' \frac{a_n i^n}{\left(\frac{n-s}{2}\right)!\left(-\frac{s+n}{2}-\rho\right)!}} \tag{208}$$

$$K_4 = \frac{i\left(\frac{c}{4}\right)^{-s-\rho-1} \sum_{n\geq s}' \frac{a_n i^{n-s}(n+\rho)}{\left(\frac{n-s}{2}\right)!\left(-\frac{s+n}{2}-\rho\right)!}}{2\sqrt{2\pi}\,(s+\rho)! \sum_{n\geq s}' \frac{a_n i^n}{\left(\frac{s-n}{2}\right)!\left(\frac{s+n}{2}+\rho\right)!}}. \tag{209}$$

When n is even let $s = 2$ for K_3 and $s = -2$ for K_4. There follows after simplification:

$$K_3' = -K_4' = K_3 \ln\frac{c}{4} + \frac{ic}{4}\sqrt{\frac{2}{\pi}}\left\{\frac{\left(C+\frac{1}{4}\right)a_2 - \sum_{n=4}^{\infty}{}' \frac{a_n}{\left(\frac{n}{2}\right)^2 - 1}}{\sum_{n=2}^{\infty}{}' a_n n}\right.$$

$$\left. - \frac{a_2 \sum_{n=2}^{\infty} n\left[a_n' + a_n\sigma\left(\frac{n}{2}\right)\right]}{\left[\sum_{n=2}^{\infty}{}' a_n n\right]^2}\right\}. \tag{210}$$

When n is odd let $s = 1$ for K_3 and $s = -1$ for K_4. We obtain:

$$K_3' = K_4' = K_3 \ln \frac{c}{4} + \frac{1}{\sqrt{2\pi}} \left\{ \frac{\left(\frac{1}{2} - C\right) a_1 + 2 \sum_{n=3}^{\infty}{}' \frac{a_n n}{n^2 - 1}}{\sum_{n=1}^{\infty}{}' a_n} + \frac{a_1 \sum_{n=1}^{\infty}{}' \left[a_n' + a_n \sigma\left(\frac{n-1}{2}\right) \right]}{\left[\sum_{n=1}^{\infty}{}' a_n \right]^2} \right\}. \tag{211}$$

In general,

$$K_3' = -(-1)^n K_4', \tag{212}$$

$$U_7 = \frac{2}{\pi} K_3' V_3 + i K_3 \frac{V_6}{\rho}. \tag{213}$$

2.13 **Recursion Formulas for a_n' when $a = -\frac{1}{2}$, $\rho \to 0$.** Let $\epsilon = b - c^2/2$. The recursion formula (100) for the Mathieu equation is then

$$a_{n+2} + a_{n-2} + \frac{4}{c^2}(\epsilon - n^2) a_n = 0. \tag{214}$$

Or, upon placing

$$r_n = a_n / a_{n-2}, \tag{215}$$

it may be expanded as in Sec. 2.6 in a continued fraction

$$r_n = \frac{-c^2/4}{\epsilon - n^2 + \frac{c^2}{4} r_{n+2}} = \cfrac{-c^2/4}{\epsilon - n^2 - \cfrac{c^4/16}{\epsilon - (n+2)^2 - \cfrac{c^4/16}{\epsilon - (n+4)^2 - \cdots}}}. \tag{216}$$

Replace n by $n + \rho$, differentiate (216) with respect to ρ, and let $\rho \to 0$.

$$r_n' = \lim_{\rho \to 0} \frac{dr_{n+\rho}}{d\rho} = -\frac{4}{c^2} r_n^2 \left(2n - \frac{c^2}{4} r_{n+2}' \right) = -\frac{4}{c^2} r_n^2 \{ 2n + r_{n+2}^2 [2(n+2) + r_{n+4}^2 (2(n+4) + \cdots)] \}. \tag{217}$$

From (215) we obtain

$$r_n' = r_n\left(\frac{a_n'}{a_n} - \frac{a_{n-2}'}{a_{n-2}}\right). \tag{218}$$

The values of r_n can be determined from (216), whence in virtue of (217) one may consider r_n' to be a known quantity. To obtain each individual a_n', one must know one particular a_n' for a specified n, or establish an additional relationship between the a_n'. There are four cases: Even functions, l even or odd; Odd functions, l even or odd.

Even functions, l and n even. According to (101) $a_n = (-1)^n a_{-n}$, and hence $a_n = a_{-n}$. Consider the coefficients to be plotted as functions of the variable $n + \rho$. In view of the symmetry about the axis $n + \rho = 0$, it is apparent that $a_0' = 0$, $a_n' = -a_{-n}'$. Since in this case a_0 differs from zero we learn from (218) that $a_2' = a_0 r_2'$. The remaining coefficients a_n' can then be found from the general formula

$$a_n' = a_n\left(\frac{a_{n-2}'}{a_{n-2}} + \frac{r_n'}{r_n}\right). \tag{219}$$

Even functions, l and n odd. Then $a_n = -a_{-n}$. The coefficient a_n is an odd function of n and its derivative is therefore even. $a_n' = a_{-n}'$. In particular $a_1' = a_{-1}'$, $a_1 = -a_{-1}$, $r_1 = -1$. From (218) we obtain $a_1' = -a_1 r_1'/2$ and the remaining a_n' are found from (219).

Odd functions, l and n odd. According to (104) $a_n = -(-1)^n a_{-n}$, and hence $a_n = a_{-n}$, $a_n' = -a_{-n}'$. From (218) one finds $a_1' = a_1 r_1'/2$ and the remaining coefficients again follow from (219).

Odd functions, l and n even. Then $a_n = -a_{-n}$, $a_n' = a_{-n}'$, $a_0 = 0$. Upon differentiating (214) and placing $n = 2$ one obtains

$$a_2' = -\frac{2\epsilon}{c^2}\, a_0'. \tag{220}$$

From (217) we find

$$\frac{r_2'}{r_2} = -\frac{4}{c^2} r_2\{4 + r_4^2[8 + \cdots]\} = -\frac{a_2}{a_0} K, \tag{221}$$

and likewise from (218) and (220)

$$\frac{r_2'}{r_2} = -\left(\frac{2\epsilon}{c^2 a_2} + \frac{1}{a_0}\right) a_0' = -\frac{a_2}{a_0} K. \tag{222}$$

Since a_2 differs from zero one may multiply (222) by a_0 to obtain

$$a_0' = a_2 K, \tag{223}$$

a finite quantity. The remaining coefficients can now be determined from (219).

In view of the symmetric or anti-symmetric relations between the coefficients the second solutions of the Mathieu equation reduce to the following:

Even functions, l and n even,

$$\frac{V_e}{\rho} = \sqrt{\frac{2}{\pi}}\left[a_0\psi + 2\sum_{n=2}^{\infty}{}' i^n(a'_n \sinh n\psi + a_n\psi \cosh n\psi)\right], \tag{224}$$

and for l and n odd,

$$\frac{V_e}{\rho} = 2\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}{}' i^n(a'_n \sinh n\psi + a_n\psi \cosh n\psi). \tag{225}$$

Odd functions, l and n even,

$$\frac{V_e}{\rho} = \sqrt{\frac{2}{\pi}}\left[a'_0 + 2\sum_{n=2}^{\infty}{}' i^n(a'_n \cosh n\psi + a_n\psi \sinh n\psi)\right], \tag{226}$$

and for l and n odd,

$$\frac{V_e}{\rho} = 2\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}{}' i^n(a'_n \cosh n\psi + a_n\psi \sinh n\psi). \tag{227}$$

The summations are extended over positive values of n. These solutions are identical with those found by Ince[10]. However, Ince's treatment expresses the coefficients a_n and a'_n as analytic functions of c and its application is limited to small values of c.

III. Prolate Spheroidal Functions

3.1 **Functions of the First Kind.** In the preceding sections, sixteen solutions have been found for the basic equation (7) and relations established that permit the analytic continuation of any one solution over the entire z-plane. We shall now specialize these functions for certain physical problems. Consider first wave functions of the prolate spheroid. In this case $a = m$, a positive integer, and the upper sign is chosen in Eqs. (3), (5), and (6). The V-functions are then functions of angle, or position on the surface of a spheroid, so that the variable is confined to the range $-1 \leq z \leq 1$. Radial variation is given by the U-functions, with the variable limited to the range $z \geq 1$. Those solutions that are finite at the poles $z = \pm 1$ are termed functions of the first kind. Functions of the second kind are infinite at these points.

For the standard V-function of the first kind we choose V_3, multiplied by a factor i^{-l}. We define:

$$V_{ml}^{(1)}(c, z) = i^{-l} V_3 = \sum_{n=0,1}^{\infty}{}' d_n^l \, T_n^m(z), \tag{228}$$

where by (92)

$$d_n^l = i^{n-l} \frac{n!}{(n+2m)!} a_n^l. \tag{229}$$

The summation starts from $n = 0$ when l is even, from $n = 1$ when l is odd, and extends over alternate positive values of n.

The coefficients will be normalized such that each spheroidal function reduces to the corresponding spherical function as $c \to 0$. It proves more convenient to carry out the normalization at $z = 0$ than at the poles $z = \pm 1$. Thus it is desired that when l is even,

$$V_{ml}^{(1)}(c, 0) = \sum_{n=0,1}^{\infty}{}' d_n^l \, T_n^m(0) = T_l^m(0), \tag{230}$$

whence

$$V_{ml}^{(1)}(0, z) = T_l^m(z), \tag{231}$$

and when l is odd,

$$\frac{d}{dz} V_{ml}^{(1)}(c, z)\Big|_{z=0} = \frac{d}{dz} T_l^m(z)\Big|_{z=0}. \tag{232}$$

For even values of l one has

$$T_l^m(0) = \frac{i^l 2^m \left(\frac{l-1}{2} + m\right)!}{\sqrt{\pi}\left(\frac{l}{2}\right)!}, \tag{233}$$

while for odd values

$$\frac{d}{dz} T_l^m(z)\Big|_{z=0} = \frac{i^{l-1} 2^{m+1} \left(\frac{l}{2} + m\right)!}{\sqrt{\pi}\left(\frac{l-1}{2}\right)!}. \tag{234}$$

Hence

$$\sum_{n=0}^{\infty}{}' \frac{\left(\frac{n-1}{2}\right)!}{\left(\frac{n}{2}+m\right)!} a_n = \frac{2^{2m}\left(\frac{l-1}{2}+m\right)!}{\left(\frac{l}{2}\right)!}, \tag{235}$$

when l is even, and

$$\sum_{n=1}^{\infty}{}' \frac{\left(\frac{n}{2}\right)!}{\left(\frac{n-1}{2}+m\right)!} a_n = \frac{2^{2m}\left(\frac{l}{2}+m\right)!}{\left(\frac{l-1}{2}\right)!} \tag{236}$$

when l is odd. This fixes the absolute magnitudes of the coefficients a_n

The corresponding solution of the *wave equation* (1) will be denoted $S_{ml}^{(1)}(c, \eta)$.

$$\begin{aligned} S_{ml}^{(1)}(c, \eta) &= (1-\eta^2)^{m/2} V_{ml}^{(1)}(c, \eta) \\ &= \sum_{n=0,1}^{\infty}{}' d_n^l P_{n+m}^m(\eta). \end{aligned} \tag{237}$$

For the standard U-function of the first kind we choose U_1 with an appropriate normalization factor.

$$\begin{aligned} U_{ml}^{(1)}(c, z) &= \kappa_l \sqrt{\frac{\pi}{2}}\, U_1 \\ &= \kappa_l (cz)^{-m} \sum_{n=0,1}^{\infty}{}' a_n^l\, j_{n+m}(cz), \end{aligned} \tag{238}$$

where κ_l is the normalization factor, and

$$j_{n+m}(cz) = \sqrt{\frac{\pi}{2cz}}\, J_{n+m+\frac{1}{2}}(cz). \tag{239}$$

The corresponding solution of the wave equation is then defined as

$$R_{ml}^{(1)}(c, \xi) = (\xi^2-1)^{m/2} U_{ml}^{(1)}(c, \xi). \tag{240}$$

When $c\xi \to \infty$,

$$j_{n+m}(c\xi) \to \frac{i^{n-l}}{c\xi} \sin\left(c\xi - \frac{l+m}{2}\pi\right), \tag{241}$$

so that

$$R_{ml}^{(1)}(c, \xi) \to \frac{\kappa_l}{c^{m+1}\xi} \sin\left(c\xi - \frac{l+m}{2}\pi\right) \sum_{n=0,1}^{\infty}{}' a_n^l\, i^{n-l}. \tag{242}$$

(Strictly (242) does not follow directly from (241), since (241) is valid only when the argument is very much larger than the order. A more careful examination of the asymptotic behavior of $a_n^l j_{n+m}(c\xi)$ on the grounds of Sec. 2.7 leads to the same result.) The normalization factor will now be chosen such that for very large values of $c\xi$,

$$R_{ml}^{(1)}(c, \xi) \to \frac{1}{c\xi} \sin\left(c\xi - \frac{l+m}{2}\pi\right). \tag{243}$$

Thus κ_l is determined by

$$\kappa_l = c^m \Big/ \sum_{n=0,1}^{\infty}{}' a_n^l i^{n-l}. \tag{244}$$

The functions $V_{ml}^{(1)}(c, z)$ and $U_{ml}^{(1)}(c, z)$ are different representations of one and the same solution. From the results of Sec. 2.9 the following relations may be established:

For l even,

$$U_{ml}^{(1)}(c, z) = \frac{\kappa_l \sqrt{\pi}\, a_0}{2^{m+1}(m+\frac{1}{2})!} \frac{V_{ml}^{(1)}(c, z)}{V_{ml}^{(1)}(c, 0)}, \tag{245}$$

and for l odd,

$$U_{ml}^{(1)}(c, z) = \frac{\kappa_l \sqrt{\pi}\, c a_1}{2^{m+2}(m+\frac{3}{2})!} \frac{V_{ml}^{(1)}(c, z)}{\left[\frac{d}{dz} V_{ml}^{(1)}(z)\right]_{z=0}}. \tag{246}$$

From the orthogonal properties of the Gegenbauer functions it may be shown that

$$\int_{-1}^{1} V_{ml}^{(1)}(c, z) V_{ml'}^{(1)}(c, z)(1-z^2)^m dz = \int_{-1}^{1} S_{ml}^{(1)}(c, \eta) S_{ml'}^{(1)}(c, \eta)\, d\eta$$

$$= \begin{cases} 0, & l \neq l' \\ 2 \sum_{n=0,1}^{\infty}{}' (a_n^l)^2 \dfrac{n!}{(n+2m)!(2n+2m+1)}, & l = l'. \end{cases} \tag{247}$$

3.2 **Functions of the Second Kind.** These are constructed from functions of the second group tabulated in Sec. 2.10. They are characterized by logarithmic singularities at the points $z = \pm 1$. We define:

$$U_{ml}^{(2)}(c, z) = \kappa_l \sqrt{\frac{\pi}{2}}\, U_5$$

$$= \kappa_l (cz)^{-m} \sum_{n=0,1}^{\infty}{}' a_n^l n_{n+m}(cz), \tag{248}$$

where $n_{n+m}(cz)$ is a spherical Bessel function of the second kind,

$$n_{n+m}(cz) = \sqrt{\frac{\pi}{2cz}}\, N_{n+m+\frac{1}{2}}(cz). \tag{249}$$

The corresponding solution of the wave equation is then

$$R_{ml}^{(2)}(c, \xi) = (\xi^2 - 1)^{m/2} U_{ml}^{(2)}(c, \xi). \tag{250}$$

As $c\xi \to \infty$,

$$R_{ml}^{(2)}(c, \xi) \to -\frac{1}{c\xi} \cos\left(c\xi - \frac{l+m}{2}\pi\right). \tag{251}$$

It is desirable to define $V_{ml}^{(2)}(c, z)$ in such a way that

$$\frac{U_{ml}^{(1)}(c, z)}{V_{ml}^{(1)}(c, z)} = \frac{U_{ml}^{(2)}(c, z)}{V_{ml}^{(2)}(c, z)} \tag{252}$$

and to obtain for it a representation valid in the neighborhood of $z = 1$. From Sec. 2.10 one finds

$$U_{ml}^{(2)}(c, z) = \kappa_l \sqrt{\frac{\pi}{2}}\, (-1)^{l+m+1} K_2 V_4. \tag{253}$$

This is combined with (252), and leads, when l is even, to

$$V_{ml}^{(2)}(c, z) = -\frac{i^{2m+1}(4m^2 - 1)}{c a_0 (\rho - 2m)!\, a_{-2m}} [V_{ml}^{(1)}(c, 0)]^2 V_4, \tag{254}$$

or

$$U_{ml}^{(2)}(c, z) = \frac{\kappa_l \sqrt{\pi}\, a_0}{2^{m+1}(m + \frac{1}{2})!} \frac{V_{ml}^{(2)}(c, z)}{V_{ml}^{(1)}(c, 0)}. \tag{255}$$

Likewise, when l is odd,

$$V_{ml}^{(2)}(c, z) = -i^{2m-1} \frac{(4m^2 - 1)(4m^2 - 9)}{c^3 a_1 (\rho + 1 - 2m)!\, a_{1-2m}} \left[\frac{d}{dz} V_{ml}^{(1)}(c, z)\right]_{z=0}^2 V_4, \tag{256}$$

and

$$U_{ml}^{(2)}(c, z) = \frac{\kappa_l \sqrt{\pi}\, c a_1}{2^{m+2}(m + \frac{1}{2})!} \frac{V_{ml}^{(2)}(c, z)}{\left[\frac{d}{dz} V_{ml}^{(1)}(c, z)\right]_{z=0}}. \tag{257}$$

The Wronskian of the two independent solutions for large values of cz reduces to

$$U_{ml}^{(1)} \frac{d}{dz} U_{ml}^{(2)} - U_{ml}^{(2)} \frac{d}{dz} U_{ml}^{(1)} = c^{-1} z^{-2m-2} \tag{258}$$

By analogy with the Bessel functions one may construct solutions corresponding to the Hankel functions.

$$U_{ml}^{(3)}(c, z) = U_{ml}^{(1)}(c, z) + iU_{ml}^{(2)}(c, z) \tag{259}$$

$$U_{ml}^{(4)}(c, z) = U_{ml}^{(1)}(c, z) - iU_{ml}^{(2)}(c, z) \tag{260}$$

$$V_{ml}^{(3)}(c, z) = V_{ml}^{(1)}(c, z) + iV_{ml}^{(2)}(c, z) \tag{261}$$

$$V_{ml}^{(4)}(c, z) = V_{ml}^{(1)}(c, z) - iV_{ml}^{(2)}(c, z). \tag{262}$$

Obviously the ratio of any $U_{ml}(c, z)$ to the $V_{ml}(c, z)$ function of the same kind and order is constant and equal to

$$\begin{cases} \dfrac{\kappa_l \sqrt{\pi}\, a_0}{2^{m+1}(m + \frac{1}{2})!\, V_{ml}^{(1)}(c, 0)} & \text{for } l \text{ even,} \\[2ex] \dfrac{\kappa_l \sqrt{\pi}\, ca_1}{2^{m+2}(m + \frac{1}{2})! \left[\dfrac{d}{dz} V_{ml}^{(1)}(c, z)\right]_{z=0}} & \text{for } l \text{ odd.} \end{cases} \tag{263}$$

The corresponding wave functions are $R_{ml}^{(3)}(c, \xi)$ and $R_{ml}^{(4)}(c, \xi)$, whose asymptotic behavior when $c\xi$ is very large is expressed by

$$R_{ml}^{(3)}(c, \xi) \to \frac{1}{c\xi} e^{i\left(c\xi - \frac{l+m+1}{2}\pi\right)}, \tag{264}$$

$$R_{ml}^{(4)}(c, \xi) \to \frac{1}{c\xi} e^{-i\left(c\xi - \frac{l+m+1}{2}\pi\right)}. \tag{265}$$

If by $R_{ml}(c, \xi)$ and $S_{ml}(c, \eta)$ written without superscript one understands any solution of the four kinds, then the prolate spheroidal wave functions of Eq. (3) are

$$W = R_{ml}(c, \xi)\, S_{ml}(c, \eta)\, e^{\pm im\varphi}. \tag{266}$$

IV. Oblate Spheroidal Functions.

4.1 **Definition.**, Let $V_{ml}(c, z)$ and $U_{ml}(c, z)$ be standard V- and U-functions as defined in III. The kind of function will normally be indicated by a superscript. Then $V_{ml}(-ic, z)$ satisfies

$$(1 - z^2)V'' - 2(m + 1)zV' + (b_l + c^2z^2)V = 0, \tag{267}$$

while $U_{ml}(-ic, iz)$ is a solution of

$$(z^2 + 1)U'' + 2(m + 1)zU' - (b_l - c^2z^2)U = 0. \tag{268}$$

These in turn are the equations of oblate spheroidal wave functions, Eqs. (3)–(6).

$$S_{ml}(-ic, \eta) = (1 - \eta^2)^{m/2} V_{ml}(-ic, \eta), \tag{269}$$

$$R_{ml}(-ic, i\xi) = i^m(\xi^2 + 1)^{m/2}\, U_{ml}(-ic, i\xi), \tag{270}$$

$$W = R_{ml}(-ic, i\xi)\, S_{ml}(-ic, \eta)\, e^{\pm i m \varphi}. \tag{271}$$

In the oblate as well as the prolate case the normalization of the coefficients a_n^l is such that in the equatorial plane transverse to the axis of revolution, $\eta = \cos\theta = 0$, the spheroidal S_{ml}-function reduces to the value of the corresponding spherical function.

V. Mathieu Functions

5.1 **Mathieu Equations.** In elliptic coordinates both the angular and radial parts of the wave function (9) satisfy an equation of the form

$$(z^2 - 1)U'' + zU' + (c^2z^2 - b)U = 0. \tag{272}$$

In the case of the angle functions z is confined to the range $|z| \leq 1$, while for the radial functions $z \geq 1$. Upon making a change of independent variable,

$$z = \cos\varphi = \cosh\psi, \qquad \psi = i\varphi, \tag{273}$$

one obtains

$$\frac{d^2V}{d\varphi^2} + (b - c^2\cos^2\varphi)V = 0, \tag{274}$$

$$\frac{d^2U}{d\psi^2} - (b - c^2\cosh^2\psi)U = 0. \tag{275}$$

Independent solutions of these equations will be defined below in a form which the authors believe to be the simplest and the most useful for application to physical problems. Apart from a slight change in the notation for the radial functions, these are the definitions proposed earlier by Stratton and Morse[6,11].

5.2 **Even Functions of the First Kind.** In this case the expansion coefficients are subject to the condition $a_n = (-1)^n a_{-n}$. Let

$$\begin{aligned} D_0 &= i\sqrt{\frac{\pi}{2}}\, a_0, \\ D_n &= i^{n+1}\sqrt{2\pi}\, a_n, \qquad (n > 0). \end{aligned} \tag{276}$$

[11] P. M. Morse, Proc. Nat. Acad. Sci., *6*, 56-62, 1935.

Then for both even and odd values of l Eqs. (172) and (173) reduce to

$$V_1 = \sum_{n=0,1}^{\infty}{}' D_n \cosh n\psi, \tag{277}$$

$$U_1 = \sqrt{\frac{2}{\pi}} \sum_{n=0,1}^{\infty}{}' i^{l-n-1} D_n J_n(cz). \tag{278}$$

We take V_1 to be the standard V- or angle function of the first kind and define:

$$Se_l^{(1)}(c, \cos\varphi) = \sum_{n=0,1}^{\infty}{}' D_n^l \cos n\varphi. \tag{279}$$

This function will be normalized at the point $\varphi = 0$, or $z = 1$, such that

$$Se_l^{(1)}(c, 1) = 1, \tag{280}$$

or

$$\sum_{n=0,1}^{\infty}{}' D_n^l = 1. \tag{281}$$

For the standard U- or radial function of the first kind we choose[12]

$$Je_l(c, z) = \kappa_l U_1(c, z), \tag{282}$$

where κ_l is a proportionality factor determined by the normalization. It is desired that as $cz \to \infty$,

$$Je_l(c, z) \to \frac{1}{\sqrt{cz}} \cos\left(cz - \frac{2l+1}{4}\pi\right). \tag{283}$$

Thus it follows that

$$\kappa_l = i^{l+1}\pi/2, \tag{284}$$

and

$$Je_l(c, z) = \sqrt{\frac{\pi}{2}} \sum_{n=0,1}^{\infty}{}' i^{l-n} D_n^l J_n(cz). \tag{285}$$

Since $U_1 = K_1 V_1$, one has

$$Je_l(c, z) = \kappa_l K_1 Se_l^{(1)}(c, z). \tag{286}$$

[12] The notation $Re_l^{(1)}$, $Re_l^{(2)}$, $Ro_l^{(1)}$, $Ro_l^{(2)}$ employed in an earlier paper (Proc. Nat. Acad. Sci., *6*, 51, 1935) for the even and odd radial functions of the first and second kinds has been replaced by Je_l, Ne_l, Jo_l, No_l to eliminate a superscript and to emphasize a relationship with the corresponding solutions of Bessel's equation.

Upon replacing a_n by D_n in (152), we obtain

$$Se_l^{(1)}(c, z) = \sqrt{2\pi}\,\lambda_l^{(e)} Je_l(c, z), \tag{287}$$

where

$$\begin{aligned} \lambda_l^{(e)} &= \frac{1}{\pi} \sum_{n=0}^{\infty}{}' i^{n-l} \frac{D_n^l}{D_0^l}, \qquad (l \text{ even}) \\ \lambda_l^{(e)} &= \frac{2}{c\pi} \sum_{n=1}^{\infty}{}' n i^{n-l} \frac{D_n^l}{D_1^l} \qquad (l \text{ odd}). \end{aligned} \tag{288}$$

5.3 **Even Functions of the Second Kind.** The radial functions of the second kind are obtained by replacing $J_n(cz)$ in (285) by the Bessel function of the second kind $N_n(cz)$.

$$\begin{aligned} Ne_l(c, z) &= \kappa_l U_5 \\ &= \sqrt{\frac{\pi}{2}} \sum_{n=0,1}^{\infty}{}' i^{l-n} D_n^l N_n(c, z). \end{aligned} \tag{289}$$

As $cz \to \infty$,

$$Ne_l(c, z) \to \frac{1}{\sqrt{cz}} \sin\left(cz - \frac{2l+1}{4}\pi\right). \tag{290}$$

The convergence of (289) is limited to the region $|z| > 1$. To obtain expressions valid in the neighborhood of $z = 1$ or $z = 0$ one must replace U_5 by its equivalent expansion (198) in terms of hyperbolic functions. We shall define

$$\begin{aligned} Se_l^{(2)}(c, z) &= U_5/K_1 \\ &= \frac{2}{\pi}\left\{\frac{K_1'}{K_1} Se_l^{(1)}(c, z) + \sum_{n=0,1}^{\infty}{}' [D_n' \sinh n\psi + \psi D_n \cosh n\psi]\right\} \end{aligned} \tag{291}$$

where

$$D_n' = i^{n+1}\sqrt{2\pi}\, a_n', \qquad (n > 0), \tag{292}$$

$$\begin{aligned} \frac{K_1'}{K_1} &= \ln\frac{c}{4} + C - \sum_{n=2}^{\infty}{}' \frac{i^n D_n}{nD_0} \\ &\quad - \frac{\sum_{n=2}^{\infty}{}' i^n \left\{D_n' + D_n\left[\sigma\left(\frac{n}{2} - 1\right) + \frac{1}{n}\right]\right\}}{\sum_{n=0}^{\infty}{}' i^n D_n} \end{aligned} \tag{293}$$

for l even, and

$$\frac{K_1'}{K_1} = ln\frac{c}{4} + C + \frac{1}{2} - 2\sum_{n=3}^{\infty}{}' \frac{i^{n-1}D_n}{(n^2-1)D_1} - \frac{\sum_{n=1}^{\infty} i^n n\left\{D_n' + D_n\left[\sigma\left(\frac{n-1}{2}\right) + \frac{1}{n}\right]\right\}}{\sum_{n=1}^{\infty} i^n n D_n} \tag{294}$$

for l odd. C and $\sigma(x)$ are defined as in (195).

It is obvious that

$$Se_l^{(3)}(c, z) = \sqrt{2\pi}\,\lambda_l^{(e)} Ne_l(c, z). \tag{295}$$

The Wronskian of $Je_l(c, \cosh\psi)$ and $Ne_l(c, \cosh\psi)$ is unity.

$$Je_l\frac{d}{d\psi}Ne_l - Ne_l\frac{d}{d\psi}Je_l = 1. \tag{296}$$

Through combinations of these functions one may define Mathieu functions of the third and fourth kinds.

$$He_l^{(1)}(c, z) = Je_l(c, z) + iNe_l(c, z), \tag{297}$$

$$He_l^{(2)}(c, z) = Je_l(c, z) - iNe_l(c, z). \tag{298}$$

The notation corresponds to that of the Hankel functions. Similar functions can be constructed from $Se_l^{(1)}(c, z)$ and $Se_l^{(2)}(c, z)$. When $cz \to \infty$,

$$He_l^{(1)}(c, z) \to \frac{1}{\sqrt{cz}}\, e^{i\left(cz - \frac{2l+1}{4}\pi\right)}, \tag{299}$$

$$He_l^{(2)}(c, z) \to \frac{1}{\sqrt{cz}}\, e^{-i\left(cz + \frac{2l+1}{4}\pi\right)}. \tag{300}$$

5.4 **Odd Functions of the First Kind.** These are constructed from U_3- and V_3-functions subject to the condition $a_n = -(-1)^n a_{-n}$. Let

$$F_n = -i^n\sqrt{2\pi}\, a_n. \tag{301}$$

Then according to (179) and (180)

$$V_3 = -i\sum_{n=1,3}^{\infty}{}' F_n \sinh n\psi, \tag{302}$$

$$U_3 = -\frac{\sqrt{z^2-1}}{cz}\sqrt{\frac{2}{\pi}}\sum_{n=1,2}^{\infty}{}' n i^{-n} F_n J_n(cz). \tag{303}$$

We take V_3 to be the standard V- or angle function of the first kind and define:

$$So_l^{(1)}(c, \cos\varphi) = \sum_{n=1,2}^{\infty}{}' F_n^l \sin n\varphi, \tag{304}$$

normalized at $z = 1$, or $\varphi = 0$, such that

$$\left[\frac{d}{d\varphi} So_l^{(1)}(c, \cos\varphi)\right]_{\varphi=0} = 1. \tag{305}$$

From this it follows that the F_n^l must be such that

$$\sum_{n=1,2}^{\infty}{}' n F_n^l = 1. \tag{306}$$

For the standard U- or radial function of the first kind we choose

$$Jo_l(c, z) = \kappa_l U_3(c, z). \tag{307}$$

The factor κ_l is fixed by the condition that when $cz \to \infty$,

$$Jo_l(c, z) \to \frac{1}{\sqrt{cz}}\cos\left(cz - \frac{2l+1}{4}\pi\right), \tag{308}$$

whence

$$\kappa_l = -i^l \pi c/2, \tag{309}$$

$$Jo_l(c, z) = \frac{\sqrt{z^2-1}}{z}\sqrt{\frac{2}{\pi}}\sum_{n=1,2}^{\infty}{}' n i^{l-n} F_n^l J_n(cz). \tag{310}$$

Since $U_3 = K_3 V_3$, one has

$$Jo_l(c, z) = \kappa_l K_3\, So_l^{(1)}(c, z). \tag{311}$$

Upon replacing a_n by F_n in (165) and (169), we obtain

$$iSo_l^{(1)}(c, z) = \sqrt{2\pi}\,\lambda_l^{(o)} Jo_l(c, z), \tag{312}$$

where

$$\begin{aligned} \lambda_l^{(o)} &= \frac{4}{\pi c^2}\sum_{n=2}^{\infty}{}' n i^{n-l}\frac{F_n}{F_2}, \qquad (l \text{ even}) \\ \lambda_l^{(o)} &= \frac{2}{\pi c}\sum_{n=1}^{\infty}{}' i^{n-l}\frac{F_n}{F_1}, \qquad (l \text{ odd}). \end{aligned} \tag{313}$$

5.5 **Odd Functions of the Second Kind.** The radial function of the second kind is defined by

$$No_l(c, z) = \kappa_l U_7 = \frac{\sqrt{z^2 - 1}}{z} \sqrt{\frac{\pi}{2}} \sum_{n=1,2}^{\infty}{}' n i^{l-n} F_n N_n(cz), \tag{314}$$

wherein U_7 is given by (181). As $cz \to \infty$,

$$No_l(c, z) \to \frac{1}{\sqrt{cz}} \sin\left(cz - \frac{2l+1}{4}\pi\right). \tag{315}$$

The convergence of (314) is rapid only when cz is large. In the neighborhood of $z = \pm 1$ one must replace this expansion by its equivalent series (213) in terms of hyperbolic functions. Again by definition,

$$So_l^{(2)}(c, z) = U_7/K_3 = \frac{2}{\pi}\left\{\frac{K_3'}{K_3} So_l^{(1)}(c, z) - i \sum_{n=0,1}^{\infty}{}' [F_n' \cosh n\psi + F_n \psi \sinh n\psi]\right\}, \tag{316}$$

where

$$F_0' = -\sqrt{\frac{\pi}{2}}\, a_0', \qquad F_n' = -\sqrt{2\pi}\, i^n a_n', \tag{317}$$

$$\frac{K_3'}{K_3} = ln\frac{c}{4} + C + \frac{1}{4} + 4\sum_{n=4}^{\infty}{}' \frac{i^n F_n}{(n^2-4)F_2} - \frac{\sum_{n=2}^{\infty}{}' n i^n \left[F_n' + F_n \sigma\left(\frac{n}{2}\right)\right]}{\sum_{n=2}^{\infty} n i^n F_n} \tag{318}$$

for l even, and

$$\frac{K_3'}{K_3} = ln\frac{c}{4} + C - \frac{1}{2} + 2\sum_{n=3}^{\infty}{}' \frac{n i^{n+1} F_n}{(n^2-1)F_1} - \frac{\sum_{n=1}^{\infty}{}' i^{n+1}\left[F_n' + F_n \sigma\left(\frac{n-1}{2}\right)\right]}{\sum_{n=1}^{\infty}{}' i^{n+1} F_n} \tag{319}$$

when l is odd. It is again apparent that

$$iSo_l^{(2)}(c, z) = \sqrt{2\pi}\, \lambda_l^{(o)} No_l(c, z). \tag{320}$$

The Wronskian of the functions $Jo_l(c, \cosh \psi)$ and $No_l(c, \cosh \psi)$ is unity.

$$Jo_l \frac{d}{d\psi} No_l - No_l \frac{d}{d\psi} Jo_l = 1. \tag{321}$$

Odd Mathieu functions of the third and fourth kinds corresponding, to the Hankel functions are constructed in the usual manner.

$$Ho_l^{(1)}(c, z) = Jo_l(c, z) + iNo_l(c, z), \tag{322}$$

$$Ho_l^{(2)}(c, z) = Jo_l(c, z) - iNo_l(c, z). \tag{323}$$

When $cz \to \infty$,

$$Ho_l^{(1)}(c, z) \to \frac{1}{\sqrt{cz}} e^{i\left(cz - \frac{2l+1}{4}\pi\right)}, \tag{324}$$

$$Ho_l^{(2)}(c, z) \to \frac{1}{\sqrt{cz}} e^{-i\left(cz - \frac{2l+1}{4}\pi\right)}. \tag{325}$$

Professor P. M. Morse and the authors are indebted to the Kennelly Fund of the Massachusetts Institute of Technology for funds to subsidize the computation of numerical values for the functions defined in this paper. In the case of the elliptic cylinder functions the coefficients D_n^l and F_n^l, and the separation constants b_l have been calculated for values of c over the range 0 to 4.5 at intervals of 0.2, and for $l = 1, 2, 3, 4$. The separation constants for both prolate and oblate spheroidal functions have been computed for values of c over the range 0 to 5.0 at intervals of 0.2, and for m and $l = 0, 1, 2, 3$; $m + l \leqq 3$. The coefficients d_m^l likewise have been calculated for the same values. These tables eventually will be published. Until then they may be obtained in mimeographed form by application to the Department of Physics at M. I. T.

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Massachusetts Institute of Technology.

A LEAST SQUARE PROCEDURE FOR SOLVING INTEGRAL EQUATIONS BY POLYNOMIAL APPROXIMATION

By F. B. Hildebrand and P. D. Crout

1. Introduction. The purpose of this paper is to continue the development of a method for solving integral equations by polynomial approximation begun in a recent paper.[1] In particular the original technique is improved using the least square method suggested in the first part of Footnote 19, the result being a least square technique which gives a very much greater accuracy with but a moderate increase in required work. This additional work, which consists essentially of the computation of additional linear equations and one extra matrix multiplication, happens to include that required for a consideration of the error; hence data concerning the error is obtained almost as a by-product.

This technique is illustrated in Section 3 by the solution of a specific potential problem—Dirichlet's problem for the cylinder for the case where the field is symmetrical about one and antisymmetrical about the other of two perpendicular axial planes, but is otherwise arbitrary. The solution obtained is of such a form that numerical results can be obtained immediately for a wide variety of fields. These results are computed in the case of two specific fields, and compare very well with the corresponding exact solutions.

The least square technique is compared with the original technique in regard to the accuracy obtained, the amount of work involved, and the amount of extra work required for a consideration of the error. Further comments will be made in Section 5.

2. Source of the Least Square Technique. If the unknown function in an integral equation is replaced by the algebraic polynomial approximation $y(x) = \sum_{i=0}^{n} K_i(\lambda) y_i$ and if the order of integration and summa-

[1] Crout, "An Application of Polynomial Approximation to the Solution of Integral Equations Arising in Physical Problems", this *Journal*, vol. 19, p. 34. Since frequent references will be made to this paper, they will be indicated by the letter a used as a suffix; thus equation (6a) and Table 3a refer to equation (6) and Table 3, respectively, of this paper.

tion is then inverted, the result is an expression of the form

$$\sum_{i=-n}^{n} y_i I_i(x) \cong \phi(x). \tag{1}$$

Here the y's are the values of $2n + 1$ equally spaced ordinates which serve to specify the polynomial, the K's are Lagrange coefficients (polynomials of degree $2n$) tables of which are available, and $\lambda = \dfrac{x - x_0}{h}$ is the distance in h units by which x lies to the right of the midpoint x_0, h being the distance between adjacent ordinates.

We shall determine the y's so that the integral of the square of the difference of the two sides of (1) is a minimum, thus

$$\int_a^b \left[\sum_{i=-n}^{n} y_i I_i(x) - \phi(x) \right]^2 dx = \text{minimum}. \tag{2}$$

Replacing the exact by an approximate integration, the integration coefficients being $D_1, D_2, \cdots, D_m$, (2) becomes

$$\sum_{k=1}^{m} D_k \left[\sum_{i=-n}^{n} y_i I_i(x_k) - \phi(x_k) \right]^2 = \text{minimum}. \tag{3}$$

Equating the corresponding partial derivatives to zero, we have

$$\sum_{k=1}^{m} D_k \left[\sum_{i=-n}^{n} y_i I_i(x_k) - \phi(x_k) \right] I_p(x_k) = 0, \qquad p = -n, \cdots, n,$$

or

$$\sum_{i=-n}^{n} y_i \sum_{k=1}^{m} D_k I_i(x_k) I_p(x_k) = \sum_{k=1}^{m} D_k \phi(x_k) I_p(x_k), \qquad p = -n, \cdots, n. \tag{4}$$

The matrix of the coefficients of the y's is hence

$$|| D_k I_p(x_k) || \cdot || I_i(x_k) ||, \tag{5}$$

where the row and column indices are p and k for the first matrix, and k and i for the second, respectively. We thus see that *the augmented matrix of the system of equations* (4) *can be obtained by forming the approximate equations*

$$\sum_{i=-n}^{n} y_i I_i(x_k) \cong \phi(x_k), \qquad k = 1, 2, \cdots, m, \tag{6}$$

in the usual manner, and then multiplying the augmented matrix[2] *of this system by the matrix obtained by multiplying the columns of the transposed (unaugmented) matrix by the integration coefficients* $D_1, D_2, \cdots, D_m$, *respectively.* It is evidently permissible to multiply all of the D's by any constant μ, since this merely multiplies both sides of the resulting linear equations (4) by μ, and hence does not alter the final results. A given set of integration coefficients may therefore be arbitrarily magnified before being used as a set of D's.

It is evident that this least square process does not require the parameters y_i in (1) to arise from the Lagrangean form of the algebraic polynomial, although the letter y and the range $-n$ to n for i were chosen with this in mind; in fact it does not even require (1) to arise from an integral equation. Any approximating function of the form $y(x) = a_1\omega_1(x) + a_2\omega_2(x) + \cdots + a_n\omega_n(x)$ could have been used.

3. Illustration of the Least Square Technique by a Potential (Dirichlet) Problem. We shall now illustrate the least square technique by solving the following problem.

An uncharged, infinite, perfectly conducting cylinder of radius R *is placed in a two dimensional electrostatic field, the field being symmetrical about one and antisymmetrical about the other of two perpendicular axial planes. Find the distribution of the induced charge.* The cylinder is shown in Fig. 1A, the planes of symmetry and antisymmetry being AC and BD, respectively. Measuring x and ξ along the circumference counterclockwise from A, it follows from symmetry that the induced surface charge density $\sigma(x)$ satisfies the relations $\sigma(-x) = \sigma(x)$ and $\sigma(\pi R - x) = -\sigma(x)$; hence $\sigma(x)$ is completely determined by its values in the first quadrant AB. Since $\sigma(x)$ is continuous at $x = \frac{\pi}{2}R$, we see that $\sigma\left(\frac{\pi}{2}R\right) = 0$.

Proceeding as in Problem 3a[1] we write the potential $\phi(x)$ due to the induced charges alone as a sum of contributions of charges contained in elementary strips of width $d\xi$, thus

$$\phi(x) = -2\int_0^{2\pi R}\left[\log\frac{r(x,\xi)}{R}\right]\sigma(\xi)\,d\xi, \tag{7}$$

[2] The "augmented matrix" differs from the matrix of the coefficients only in that it contains an additional column composed of the right hand sides of the expressions. The first of the approximate linear equations corresponding to (9) is thus $12.17192y_{-2} + 27.39411y_{-1} + 1.98274y_0 + 7.37574y_1 \cong 6.63553\frac{\phi_1}{R}$.

where $r(x, \xi)$ is the length of the chord joining points x and ξ in Fig. 1A, and where the axis of the cylinder is at zero potential. Next we approximate $\sigma(x)$ in the first quadrant AB by a fourth degree polynomial $y(x)$, whereupon $\sigma(x)$ is automatically approximated in the other quadrants by separate but corresponding polynomials, the values of which at intervals of $\frac{\pi}{8} R$ are shown in Fig. 1A. For any x we have

$$\sigma(x) \cong y(x) = \sum_{i=-2}^{1} K_i(\lambda) y_i, \tag{8}$$

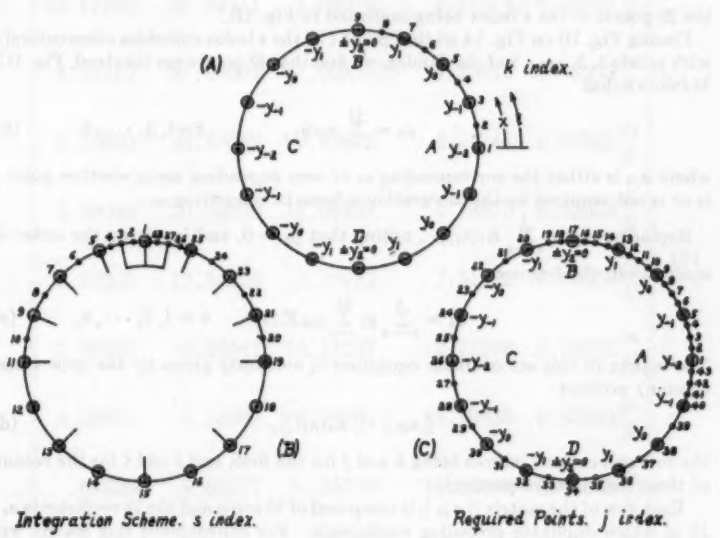

FIG. 1

where $h = \frac{\pi}{8} R$, and λ is the distance in h units between x and the center of the quadrant which contains x, which quadrant must be noted in choosing the K's and y's.

Using (7) with $\sigma(x)$ replaced by $y(x)$, the integral being evaluated by an appropriate scheme of approximate integration,[3] we find that the

[3] The integration scheme used is shown in Fig. 1B, the potential at point $s = 1$ being obtained from (7) by breaking the interval of integration up into fourteen subintervals as indicated. The three point integration formula ((11a)

corresponding approximate values of $\phi(x)$ at points $k = 1, 2, \cdots, 8$, Fig. 1A, are given by a set of linear equations whose augmented matrix is (9).[4]

---

with $n = 1$) is applied to each of these except the two which are adjacent to point 1. These two are treated alike, the interval bounded by 1 and 3 being taken care of by replacing $y(x)$ by a second degree polynomial which agrees with it at 1, 2, and 3; replacing the chord r by the arc length measured from 1; and actually carrying out the integration. The result is an expression of the form

$$\phi_1 = \sum_{s=1}^{28} a_s' y_s , \tag{a}$$

the 28 points of the s index being indicated in Fig. 1B.

Placing Fig. 1B on Fig. 1A so that point 1 of the s index coincides consecutively with points 1, 2, $\cdots$, 8 of the k index, we find that 43 points are involved, Fig. 1C. It follows that

$$\phi_k = \sum_{j=1}^{43} a_{jk} y_j , \qquad k=1, 2, \cdots, 8, \tag{b}$$

where a_{jk} is either the corresponding a_s or zero depending upon whether point j is or is not required by the integration scheme in computing ϕ_k.

Replacing y_j by $\sum_{i=-2}^{1} K_i(\lambda_j) y_i$, noting that $y_2 = 0$, and inverting the order of summation, (b) becomes

$$\phi_k = \sum_{i=-2}^{1} y_i \sum_{j=1}^{43} a_{jk} K_i(\lambda_j), \quad k = 1, 2, \cdots, 8. \tag{c}$$

The matrix of this set of linear equations is evidently given by the (row times column) product

$$\| a_{jk} \| \cdot \| K_i(\lambda_j) \| , \tag{d}$$

the row and column indices being k and j for the first, and j and i for the second of these matrices, respectively.

Each row of the matrix $\| a_{jk} \|$ is composed of 15 zeros and the 28 coefficients a_s, 13 of which duplicate preceding coefficients. For convenience this matrix was written with the 8 rows and 43 columns interchanged, and labelled "transposed". The matrix $\| K_i(\lambda_j) \|$ was taken directly from Tables 3a and 6. (See Problem 3a.) The product matrix is $\frac{1}{6.63553}$ times the matrix formed by the first four columns of (9).[4]

[4] The data given here (except where exact) is obtained by omitting a certain number of decimal places from the corresponding original data, which was obtained using a ten bank computing machine retaining ten decimal places throughout, and using seven place tables of common logarithms and seven place tables of trigonometric functions. Although the error introduced by the computation was not separately considered, it seems evident that the number of significant figures obtained is more than sufficient for our purposes.

If $\sigma(x)$ were actually a fourth degree polynomial in the first quadrant AB, in which case $\sigma(x)$ would be given by corresponding polynomials in the other quadrants, then the equations corresponding to (9) would be exact instead of approximate except for small errors introduced by the approximate integrations; however, $\sigma(x)$ is not a polynomial, and our problem is to determine the four y's of the i index so that for a given set of ϕ's all eight of these equations will be satisfied closely, though none will be satisfied exactly except by accident. Using the least square procedure obtained in Section 2, we proceed as follows.

$$\left\|\begin{array}{rrrrr}
12.17192 & 27.39411 & 1.98274 & 7.37574 & 6.63553\frac{\phi_1}{R} \\
8.29142 & 31.65904 & -.0037076 & 8.79647 & 6.63553\frac{\phi_2}{R} \\
8.11630 & 23.33852 & 9.83972 & 4.94741 & 6.63553\frac{\phi_3}{R} \\
3.99095 & 20.54120 & 13.99221 & 4.73875 & 6.63553\frac{\phi_4}{R} \\
3.07058 & 13.54339 & 15.59732 & 7.51721 & 6.63553\frac{\phi_5}{R} \\
1.59243 & 10.32541 & 10.11027 & 12.77396 & 6.63553\frac{\phi_6}{R} \\
3.05811 & 3.15097 & 6.98407 & 13.19839 & 6.63553\frac{\phi_7}{R} \\
.043290 & 4.66271 & -.25749 & 10.36608 & 6.63553\frac{\phi_8}{R}
\end{array}\right\| \qquad (9)$$

The quantities D_k are, except for an arbitrary constant multiplier, the coefficients in an approximate integration formula for nine points: the eight points of the k index, and the point $x = \frac{\pi}{2}R$. The equation corresponding to the latter point has been neglected since it is merely $0 = 0$ due to symmetry and the fact that BD is a plane of zero potential. A consideration of this equation would have led to the addition of a row of zeros at the bottom of (9). Choosing Simpson's rule as the source of the D's, reserving all comment for Section 5, we have

$$D_1 = \tfrac{1}{2}, \quad D_2 = D_4 = D_6 = D_8 = 2, \quad D_3 = D_5 = D_7 = 1. \qquad (10)$$

The final $D_9 = \frac{1}{2}$, which would be associated with the equation $0 = 0$, is omitted. We now drop the last column of (9), multiply each row by the corresponding D from (10) (k^{th} row multiplied by D_k), and finally interchange the rows and columns of the resulting matrix, thereby ob-

Transposed

$$\begin{Vmatrix} 6.08596 & 13.69705 & .99137 & 3.68787 \\ 16.58284 & 63.31809 & -.0074153 & 17.59295 \\ 8.11630 & 23.33852 & 9.83972 & 4.94741 \\ 7.98190 & 41.08239 & 27.98441 & 9.47749 \\ 3.07058 & 13.54339 & 15.59732 & 7.51721 \\ 3.18486 & 20.65082 & 20.22055 & 25.54791 \\ 3.05811 & 3.15097 & 6.98407 & 13.19839 \\ .086581 & 9.32543 & -.51498 & 20.73216 \end{Vmatrix} \tag{11}$$

$$\begin{Vmatrix} 333.2 & 1129.6 & 305.0 & 373.8 & 6.636\,(6.086\,\phi_1/R + 16.583\,\phi_2/R + 8.116\,\phi_3/R + 7.982\,\phi_4/R + 3.071\,\phi_5/R + 3.185\,\phi_6/R + 3.058\,\phi_7/R + .0866\,\phi_8/R) \\ 1129.6 & 4218.4 & 1271.0 & 1472.0 & 6.636\,(13.697\,\phi_1/R + 63.318\,\phi_2/R + 23.339\,\phi_3/R + 41.082\,\phi_4/R + 13.543\,\phi_5/R + 20.651\,\phi_6/R + 3.151\,\phi_7/R + 9.325\,\phi_8/R) \\ 305.0 & 1271.0 & 987.0 & 650.9 & 6.636\,(.991\,\phi_1/R - .00742\,\phi_2/R + 9.840\,\phi_3/R + 27.984\,\phi_4/R + 15.597\,\phi_5/R + 20.221\,\phi_6/R + 6.984\,\phi_7/R - .515\,\phi_8/R) \\ 373.8 & 1472.0 & 650.9 & 1023.3 & 6.636\,(3.688\,\phi_1/R + 17.593\,\phi_2/R + 4.947\,\phi_3/R + 9.477\,\phi_4/R + 7.517\,\phi_5/R + 25.548\,\phi_6/R + 13.198\,\phi_7/R + 20.732\,\phi_8/R) \end{Vmatrix} \tag{12}$$

taining the "derived matrix" (11). Here the interchange of rows and columns is merely indicated by the word "transposed".[5] The (row times

[5] This transposition can evidently be omitted if the following matrix multiplication is made column times column instead of row times column. Since in (11) the transposition is only indicated, this is actually what was done.

column) product of the matrix (11) times the matrix (9) gives the augmented matrix (12) of the final set of linear equations.[2] It may be noted that each element of the last column, which is the right hand side of the corresponding equation, consists of the sum of eight terms the coefficients of which make up the corresponding row in (11); in fact the matrix of the coefficients of the ϕ's in (12) is identical with (11).

The solution of the linear equations may be carried out in a manner similar to that used on the system (54a), the same abbreviated notation being used for the ϕ columns as for the y columns.[6] Each ϕ column is treated exactly the same as the β' column of (54a), and remains independent of the other ϕ columns throughout the solution. The final result is given by (13).[4]

$$
\begin{aligned}
y_{-2} &= .52117\frac{\phi_1}{R} - .33858\frac{\phi_2}{R} + .45551\frac{\phi_3}{R} - .53594\frac{\phi_4}{R} + .022517\frac{\phi_5}{R} \\
&\qquad - .29627\frac{\phi_6}{R} + .64303\frac{\phi_7}{R} - .52588\frac{\phi_8}{R} \\
y_{-1} &= -.11918\frac{\phi_1}{R} + .25335\frac{\phi_2}{R} - .089247\frac{\phi_3}{R} + .20637\frac{\phi_4}{R} - .040027\frac{\phi_5}{R} \\
&\qquad + .021795\frac{\phi_6}{R} - .23878\frac{\phi_7}{R} + .10903\frac{\phi_8}{R} \\
y_0 &= -.0072184\frac{\phi_1}{R} - .23797\frac{\phi_2}{R} + .076198\frac{\phi_3}{R} + .19662\frac{\phi_4}{R} + .13374\frac{\phi_5}{R} \\
&\qquad + .068010\frac{\phi_6}{R} + .047690\frac{\phi_7}{R} - .16069\frac{\phi_8}{R} \\
y_1 &= .0095933\frac{\phi_1}{R} + .024676\frac{\phi_2}{R} - .054385\frac{\phi_3}{R} - .16472\frac{\phi_4}{R} - .017181\frac{\phi_5}{R} \\
&\qquad + .19926\frac{\phi_6}{R} + .16386\frac{\phi_7}{R} + .27189\frac{\phi_8}{R}.
\end{aligned}
\tag{13}
$$

[4] A method for solving systems of linear algebraic equations has been devised which is very much simpler than Gauss' method, used above. Since the work involved using this method is still further cut almost in half if the coefficients of the unknowns are symmetrical about the principal diagonal, as is always the case with the least square procedure, it follows that the work required to solve the linear equations is in general less with the least square technique than with the original technique (depending upon the number of ϕ columns). Crout, "A Short Method for Evaluating Determinants, and Solving Systems of Linear Equations with Real or Complex Coefficients", *Transactions of the American Institute of Electrical Engineers*, August 1941.

The condition for the cylinder to be an equipotential surface is that $\phi(x)$ be the negative of the potential due to all *except* the induced charges. If $\phi(x)$ is known, (7) is an integral equation of the first kind, the solution of which is given by (13). Such is the case if the remaining charges are fixed in space, or if the field is such that the presence of the induced charges does not appreciably alter the field due to the remaining charges (by altering their distribution).

4. Applications. Let us apply the results (13) to the following two problems.

1. Find the induced charge distribution if the cylinder is placed in a field originally uniform and of intensity E. The potential at the surface of the cylinder due to the original field alone is $-ER\cos\frac{x}{R}$; hence placing $\phi_k = ER\cos\frac{x_k}{R}$ in (13), we have

$$\begin{gathered} y_{-2} = .15912\,E, \qquad y_{-1} = .14699\,E, \qquad y_0 = .11252\,E, \\ y_1 = .06089\,E, \qquad y_2 = 0. \end{gathered} \tag{14}$$

These values give a rough picture of $y(x)$. Other values can be obtained using these in connection with (6a) and Table 3a.[1] A comparison of these results with the known exact solution is given in Table 1, from which we see that if the approximate and exact solutions were plotted on a full page curve sheet, the two curves would coincide in so far as the eye could see.

2. Find the induced charge distribution due to two oppositely charged wires on opposite sides of the cylinder, both parallel to the axis and distant 2R from it, the charge per unit length being q and −q, respectively. The negative wire is to the right of A, and the positive wire is to the left of C on Fig. 1A. Noting that the center of the cylinder is a point of zero potential we compute the negative of the potential at each point of the k index due to the two wires alone. This computation is simple using the law of cosines to obtain the required distances, for the radical can be replaced by a $\frac{1}{2}$ before the logarithm. The result is

$$\begin{gathered} \phi_1 = 2.1972\,q, \qquad \phi_2 = 2.1146\,q, \qquad \phi_3 = 1.8970\,q, \\ \phi_4 = 1.6041\,q, \qquad \phi_5 = 1.2823\,q, \qquad \phi_6 = .95554\,q, \\ \phi_7 = .63258\,q, \qquad \phi_8 = .31472\,q. \end{gathered} \tag{15}$$

Placing these values in (13) we have finally

$$y_{-2} = .42062\,\frac{q}{R}, \qquad y_{-1} = .31529\,\frac{q}{R}, \qquad y_0 = .15694\,\frac{q}{R},$$
$$y_1 = .06346\,\frac{q}{R}, \qquad y_2 = 0. \tag{16}$$

Using these with (6a) and Table 3a we obtain the other values of $y(x)$ given in Table 2.

The exact solution was obtained using images,[7] the cylinder being replaced by two wires carrying charges of q and $-q$ per unit length, respectively, both parallel to and on opposite sides of the axis, and distant $\frac{1}{2}R$ from it. The four wires and the axis all lie in a plane. $\sigma(x)$ was obtained by dividing the radial component of field intensity at the points occupied by the surface of the cylinder (before it was removed) by 4π. Values of $\sigma(x)$ are given in Table 2; also the approximate and exact solutions are shown by full lines in Fig. 2.

5. Critical Examination of the Least Square Technique. In this section certain general comments will be made concerning the least square process itself, and the comparative accuracy and difficulty of the least square technique and the original technique.

A. Choice of the Integration Coefficients D_k. In the integral equation (7) $\phi(x)$ is given, and the four y's of the i index, which merely serve to specify the polynomial, are to be determined so that the two sides of the expression

$$\phi(x) \cong -2\int_0^{2\pi R}\left[\log\frac{r(x,\xi)}{R}\right]y(\xi)\,d\xi \tag{17}$$

approximate each other over the interval $0 \leq x \leq \frac{\pi R}{2}$. Placing

$$\Phi(x) = \phi(x) + 2\int_0^{2\pi R}\left[\log\frac{r(x,\xi)}{R}\right]y(\xi)\,d\xi, \tag{18}$$

we see that (17) is merely

$$\Phi(x) \cong 0. \tag{19}$$

[7] It can be shown that if two infinite wires carrying charges of equal magnitude but opposite sign are placed parallel to the axis of a cylinder of radius R, in the same axial plane, and at distances $\frac{R}{\mu}$ and μR from the axis on the same side, then the cylinder is an equipotential surface for any μ.

Although the results obtained in Section 2 were obtained in an effort to minimize the integral $\int_0^{\frac{\pi}{2}R} [\Phi(x)]^2\,dx$,[8] the expression actually minimized

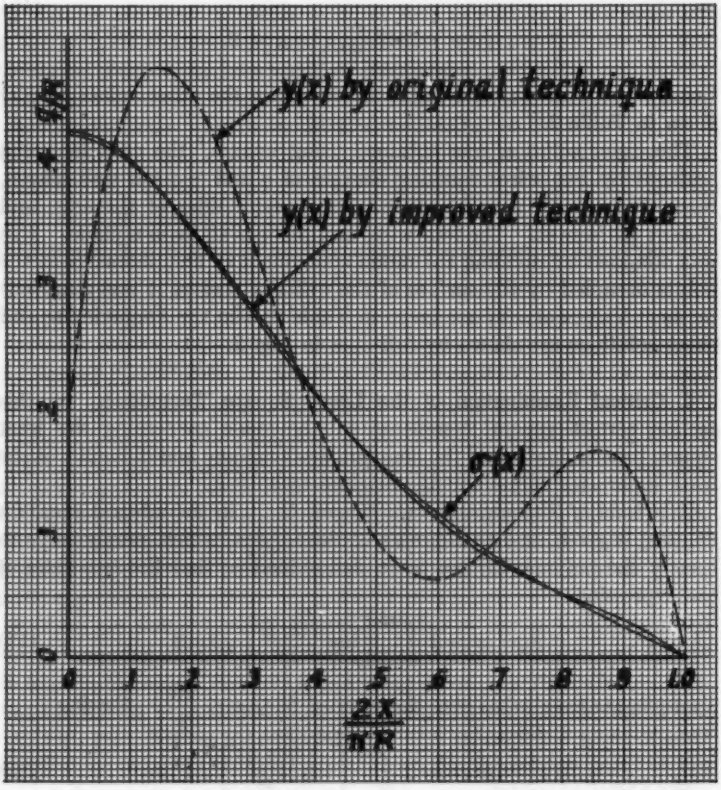

FIG. 2

is $\sum_{k=1}^{6} D_k[\Phi(x_k)]^2$. Since the curve of $\Phi(x)$ probably crosses the x axis several times (with the former technique the y's are determined so that $\Phi(x) = 0$ at five equally spaced points), and since the curve of $[\Phi(x)]^2$ is

[8] The bracket in (3) is $-\Phi(x_k)$ except for small errors introduced by the approximate integrations leading to (b);[3] hence this bracket squared is $[\Phi(x_k)]^2$.

more wavy than that of $\Phi(x)$ $\left(\text{since } \frac{d}{dx}[\Phi(x)]^2 = 0 \text{ not only when } \frac{d}{dx}\Phi(x) = 0 \text{ but also when } \Phi(x) = 0\right)$, the weighted sum $\sum_{k=1}^{8} D_k[\Phi(x_k)]^2$ cannot be expected to give a very close approximation of the integral unless more points are used with the k index. Nevertheless the minimization of the weighted sum requires the curve of $\Phi(x)$ to be drawn down to the x axis at the eight points of the k index, the relative "tightness" of the drawing being given by the D's, and should hence in itself be capable of forcing this curve to approximate the x axis, as required by (19).

If there be associated with each point of the k index the interval of the x axis between the midpoints of the two subintervals which meet at that point, and if the corresponding weighting factors D_k are made proportional to these associated intervals, we have

$$D_1 = \tfrac{1}{2}, \quad D_2 = D_3 = D_4 = \cdots = D_8 = 1, \quad D_9 = \tfrac{1}{2}, \tag{20}$$

which also happen to be the D's associated with the trapezoidal integration formula.[9] This set of values is extremely simple to use because of the 1's; for example if it had been used in place of (10) in Section 3, the matrix (11) would be obtained from (9) by omitting the last column and dividing the elements of the first row by 2. Since the values (20) are easy to apply, represent weightings proportional to the associated intervals and hence appear reasonable intuitively regardless of any consideration of $\int_0^{\frac{\pi}{2}R} [\Phi(x)]^2\,dx$, and arise from an integration formula which though crude does not have to be accurate since a good approximation to this integral is too much to expect, it appears that (20) should be a good set of D's to use in practice. As a matter of fact very good results have been obtained using (20) in several different problems. Certain of these results will be given in Part C of this section.

On the basis of the above discussion the D's of (20) would be finally chosen except for one fact—the better integration formulas (for equally spaced points) all have comparatively large coefficients associated with alternate ordinates beginning with the second, and comparatively small coefficients associated with the first and last ordinates. For example, noting that the integration coefficients may all be multiplied by any

[9] Here the value $k = 9$ referring to $x = \frac{\pi}{2}R$ is added for completeness, although $\Phi(x_9) = 0$ and the corresponding equation $0 = 0$ was omitted in Section 3.

constant, we have the following sets of D's arising from several different integration formulas.

Formula	D_{-4}	D_{-3}	D_{-2}	D_{-1}	D_0	D_1	D_2	D_3	D_4
3A				.500	2.000	.500			
5A			.467	2.133	.800	2.133	.467		
5B			.500	2.000	1.000	2.000	.500		
7A		.439	2.314	.289	2.914	.289	2.314	.439	
7B		.500	2.000	1.000	2.000	1.000	2.000	.500	
9A	.419	2.492	−.393	4.443	−1.922	4.443	−.393	2.492	.419
9B	.500	2.000	1.000	2.000	1.000	2.000	1.000	2.000	.500
9C	.467	2.133	.800	2.133	.933	2.133	.800	2.133	.467

Here subscripts refer to corresponding ordinates. In the first column the number indicates the number of points involved, and the letter, the formula. A refers to Cotes' formula (11a); B, to Simpson's rule; and C, to the formula obtained by using Cotes' five point formula twice. For comparison the coefficients have been magnified so that the sums are the same for all sets involving the same number of points. In spite of the fact that $\sum_{k=1}^{9} D_k[\Phi(x_k)]^2$ may not give a good approximation of the integral, the above situation indicates that the first and last D's should be comparatively small, and the alternate D's beginning with the second should be comparatively large.[9] Accordingly the coefficients

$$D_1 = \tfrac{1}{2}, \quad D_2 = 2, \quad D_3 = 1, \quad D_4 = 2, \quad D_5 = 1, \quad D_6 = 2, \quad D_7 = 1, \quad D_8 = 2, \quad D_9 = \tfrac{1}{2}, \tag{21}$$

obtained from Simpson's rule were finally chosen, and are recommended for the following reasons.[9,10]

[10] The solutions obtained using the least square technique with $D_1 = D_3 = D_5 = D_7 = 0$, $D_2 = D_4 = D_6 = D_8 = 1$ and with $D_1 = D_3 = D_5 = D_7 = 1$, $D_2 = D_4 = D_6 = D_8 = 0$ are given in Tables 1, 2, 3, 4; these evidently coinciding with those obtained using the original technique and solving the four equations corresponding to $k = 2, 4, 6, 8$, or $k = 1, 3, 5, 7$, respectively. Since these two sets of D's represent extremes of favoring or neglecting the coefficients associated with points $k = 2, 4, 6, 8$, which are favored by the integration formulas, and since the first set gives much better results than the second, we have another indication that D_2, D_4, D_6, D_8 should be larger than D_1, D_3, D_5, D_7, and hence that the coefficients (21) are preferable to (20).

1. Although these D's fluctuate in the way desired, these fluctuations are milder than those indicated by the other integration formulas. For example some of the unfavored coefficients in Cotes' nine point formula are actually negative. We are thus deviating no further from the intuitive picture which gave (20) than is suggested by a good integration formula.

2. These D's are easy to apply because half of them indicate no alteration of rows, and the other half indicate either multiplication or division by 2.

Although the coefficients (21) seem somewhat preferable to (20), two things indicate that the results obtained in the two cases will differ but little. First, in a term $D_k[\Phi(x_k)]^2$ a multiplication (or division) of D_k by 2 could be compensated by dividing (or multiplying) $\Phi(x_k)$ by only $\sqrt{2}$. Should any one $\Phi(x_k)$ try to become appreciably larger than the others, its term would, because of the square, immediately monopolize the sum $\sum_{k=1}^{9} D_k[\Phi(x_k)]^2$ regardless of whether D_k is $\frac{1}{2}$, 1, or 2. Second, it seems reasonable that (20) and (21) are both capable of obtaining about all that can be hoped for from a fourth degree polynomial approximation. For comparisons in specific cases see Part C of this section.

B. Comparison of the Amounts of Work Required by the Original Technique and the Least Square Technique. The theoretical work involved in setting up a problem is the same for both techniques. In actual work of computation the new technique differs from the old in that more linear equations must be computed and an additional matrix multiplication carried out. More specifically, had the original technique been used in Section 3 and Footnote 3 the work of obtaining the two matrices in (d) would be almost exactly the same, although the first (transposed) matrix would have four instead of eight columns, and the second matrix would have 37 instead of 43 rows. The work involved in the matrix multiplication leading to (9) would be just half that required above; and (9) would consist of the first, third, fifth, and seventh rows of (9) above, which rows express the four linear equations which would be solved. The formulation of (11) and the matrix multiplication required to obtain (12) would be avoided altogether. The solution of the linear equations would be somewhat though not much simpler, since four instead of eight ϕ columns would be carried on the right hand side.[6]

Actually the work of formulating (11) is slight, and that required by the matrix multiplication leading to (12) is small; furthermore the addi-

tional work involved in solving the linear equations is small. The large part of the difference in the work required by the two techniques lies in the greater amount of work required in the matrix multiplication (d) by the least square technique.

In the comparison just made it was assumed that no attempt to investigate the error is associated with the old technique. Such an attempt would require the computation of additional linear equations, as described in Section 6, and would hence remove the main difference in the amounts of work required by the two techniques.

C. Comparison of the Accuracies Associated with the Original Technique and the Least Square Technique. The solution obtained by solving the system comprising the first, third, fifth, and seventh equations of (9) using the original technique is given by (22).[4]

$$\begin{aligned} y_{-2} &= -9.18716\frac{\phi_1}{R} + 17.05345\frac{\phi_3}{R} - 12.11730\frac{\phi_5}{R} + 5.64312\frac{\phi_7}{R} \\ y_{-1} &= 3.79976\frac{\phi_1}{R} - 6.60654\frac{\phi_3}{R} + 4.73403\frac{\phi_5}{R} - 2.34328\frac{\phi_7}{R} \\ y_0 &= -2.79137\frac{\phi_1}{R} + 4.72977\frac{\phi_3}{R} - 2.82987\frac{\phi_5}{R} + 1.39874\frac{\phi_7}{R} \\ y_1 &= 2.69863\frac{\phi_1}{R} - 4.87691\frac{\phi_3}{R} + 3.17488\frac{\phi_5}{R} - .98550\frac{\phi_7}{R}. \end{aligned} \tag{22}$$

The solution obtained using the least square technique with the coefficients (20) instead of (10) or (21) is given by (23).[4] It is interesting to compare the coefficients in (23) with those in (13).

Tables 1 and 2 give corresponding values of the exact solutions of Problems 1 and 2, Section 4, and the approximate solutions obtained from (13), (22), and (23), respectively. We see that if the solution given by (23) for Problem 2 were plotted on Fig. 2, it could not be distinguished by eye from the curve $y(x)$ obtained from (13). We also see that whereas good results are obtained for Problem 2 using the least square technique with either (20) or (21) as D's, the original technique fails completely.

To permit further comparisons in the case of more rapidly fluctuating distributions of potential and induced charge, the solutions given by (13), (22), and (23) were obtained corresponding to $\phi(x) = L\cos\frac{3x}{R}$ and $\phi(x) = L\cos\frac{5x}{R}$, respectively, and values given in Tables 3 and 4

beside those of the corresponding exact solutions $\sigma(x) = \frac{3L}{2\pi R}\cos\frac{3x}{R}$ and $\sigma(x) = \frac{5L}{2\pi R}\cos\frac{5x}{R}$ obtained by Fourier methods. It is not surprising that comparatively poor approximations are obtained in the last case, because the fourth degree polynomial cannot follow the rapid fluctuations of $\sigma(x)$ over such a long interval.

It may be noted that the solutions given by (13) differ but little from those given by (23), and are slightly better, which fact is in line with the prediction given in Part A.

$$
\begin{aligned}
y_{-2} &= .55591\frac{\phi_1}{R} - .40498\frac{\phi_2}{R} + .44918\frac{\phi_3}{R} - .41650\frac{\phi_4}{R} - .064474\frac{\phi_5}{R} \\
&\quad - .33090\frac{\phi_6}{R} + .65002\frac{\phi_7}{R} - .44897\frac{\phi_8}{R} \\
y_{-1} &= -.11591\frac{\phi_1}{R} + .24818\frac{\phi_2}{R} - .069020\frac{\phi_3}{R} + .16337\frac{\phi_4}{R} - .00047962\frac{\phi_5}{R} \\
&\quad + .054665\frac{\phi_6}{R} - .27109\frac{\phi_7}{R} + .10828\frac{\phi_8}{R} \\
y_0 &= -.029229\frac{\phi_1}{R} - .21726\frac{\phi_2}{R} + .098764\frac{\phi_3}{R} + .14344\frac{\phi_4}{R} + .19549\frac{\phi_5}{R} \\
&\quad + .030191\frac{\phi_6}{R} + .046115\frac{\phi_7}{R} - .14524\frac{\phi_8}{R} \\
y_1 &= -.0017367\frac{\phi_1}{R} + .055664\frac{\phi_2}{R} - .10186\frac{\phi_3}{R} - .11511\frac{\phi_4}{R} - .034333\frac{\phi_5}{R} \\
&\quad + .16430\frac{\phi_6}{R} + .22213\frac{\phi_7}{R} + .23496\frac{\phi_8}{R}
\end{aligned}
\tag{23}
$$

6. Consideration of the Error. It was pointed out in Section 5a[1] that if we place

$$f(x) = y(x) + \epsilon(x) \text{ and } \Phi(x) = \phi(x) - \int_a^b G(x, \xi)y(\xi)\,d\xi, \tag{24}$$

the integral equation

$$\phi(x) = \int_a^b G(x, \xi)f(\xi)\,d\xi \tag{25}$$

becomes

$$\Phi(x) = \int_a^b G(x, \xi)\epsilon(\xi)\,d\xi, \tag{26}$$

an integral equation for the error $\epsilon(x)$. A comparison of the known functions $\Phi(x)$ and $\phi(x)$ together with a physical interpretation of (26) would in general give a pretty definite idea as to the importance of $\epsilon(x)$. Thus if for $a \leq x \leq b$ $|\Phi(x)|$ is on the whole small compared with $|\phi(x)|$, we should expect $\epsilon(x)$ to be on the whole small in comparison with $f(x)$ because of the similarity of (25) and (26). Physically if (25) arises from a potential problem, $\epsilon(x)$ is the charge distribution corresponding to the small potential distribution $\Phi(x)$.

Although $\phi(x)$ is given, $\Phi(x)$ must be determined point by point. Since $\Phi(x_k)$ is the difference between the two sides of the (approximate) linear equation corresponding to point x_k with the y's replaced by values of the approximate solution $y(x)$, we see that each value of Φ requires the determination of a linear equation. With the original technique the Φ's corresponding to the computed equations are zero; hence a consideration of the error would require the computation of additional equations. With the least square technique, however, the greater number of equations required by the least square process is available for determining a correspondingly larger number of Φ's, this number being in general sufficient to give a satisfactory picture of $\Phi(x)$ without the determination of additional equations. We thus see that *whereas with the original technique a consideration of the error is in general not practicable, with the least square technique such a consideration requires only a short final calculation.*

Following the above process the right hand sides ϕ_k and the right minus the left hand sides Φ_k of the approximate equations (9) are given in Tables 1, 2, 3, 4 for the corresponding problems. Values of Φ_k are also tabulated for the other approximate solutions.

7. Extensions of the Least Square Technique. Although the discussion in the preceding sections was associated with certain specific problems, the following generalizations are self-evident.

A. If the number of points of the k index is the same as the number of parameters which fix the polynomial, the results given by the least square technique coincide with those given by the original technique. The number of points must hence be greater than the number of parameters; however, there is no other restriction on the number or spacing of these points.[11] Enough should be used, however, to justify

[11] This breaking of the equality between the number of points of the k index and the number of independent parameters (or y's), which was part of the original

the additional work required by the least square technique; and the distribution and weighting coefficients should be reasonable. Unless there is reason for doing otherwise, it is recommended that the points of the k index divide the interval into an even number of equal subintervals, and the coefficients D_k be given the values $\frac{1}{2}, 2, 1, 2, 1, \cdots, 1, 2, \frac{1}{2}$, corresponding to Simpson's rule (although the results thus obtained will probably not differ much from those obtained using the coefficients $\frac{1}{2}, 1, 1, 1, \cdots, 1, 1, \frac{1}{2}$). However, in certain problems other spacings or weightings may be suggested by the nature of the problem itself.[12]

B. The use of the Lagrangean form $\sum_{i=-n}^{n} K_i(\lambda) y_i$ is entirely a matter of convenience. The same results would be obtained regardless of what parameters are used to specify the polynomial.

The various processes which compose the least square technique can be carried out with any approximating function of the form $y(x) = a_1\omega_1(x) + a_2\omega_2(x) + \cdots + a_n\omega_n(x)$, and hence do not require this function to be an algebraic polynomial.

The entire discussion contained in Parts A, B, C, D, F, G, H of Section 4a[1] applies equally well to the least square technique; hence no further comments in this direction will be made here.

C. The least square technique as described can be applied simply and effectively to integral equations of the first kind and nonhomogeneous

---

technique, is particularly desirable if restrictions are imposed on $y(x)$ so that the parameters are not all independent. For example in Problem 4a[1] when the restriction (95a) was imposed to make $y'(0)$ vanish, the point $k = -1$ was omitted so that the number of points of the k index would remain equal to the number of independent y's. The result was the poorer of the two approximate solutions, the error being worst in the vicinity of the omitted point. Much better results would be obtained by retaining this point and using the least square technique.

If restrictions are imposed on $y(x)$, the approximate equations corresponding to the points of the k index are first obtained, after which the equations of restriction are used to eliminate a corresponding number of arbitrarily chosen y's (or parameters). After this the least square technique is used as before. Since the equations of restriction are linear in the y's (or parameters), the elimination process merely consists of the first few steps in solving a set of linear equations.

[12] The field of flow around a body of revolution such as an airship can be found by placing a polynomial distribution of fictitious sources and sinks along the axis. In this case the integral equation states that the component of flow perpendicular to the surface of the ship is zero. The points of the i index lie along the axis, and the points of the k index lie on the curve which represents the surface on an axial section.

integral equations of the second kind. It can also be applied to other varieties of integral equations, perhaps with certain obvious modifications.

It can also be applied to homogeneous integral equations of the second kind; in fact this has been done with very good results.[11] Although the application to homogeneous equations is simplified by the fact that the ϕ's are all zero, it is complicated by the fact that the extra matrix multiplication introduced by the least square process doubles the degree of the characteristic equation, which states that the determinant of the resulting set of homogeneous linear equations vanishes.

8. Failure of the Original Technique When Applied to Problem 2. The problem considered in Section 3, which is an extension of Problem 3a,[1] was solved using the original technique, and the solution (22) obtained before there was any indication that an improvement in this technique was necessary. Applied to a uniform field, (22) gave good results, as is seen in Table 1. Problem 2 was devised primarily to give (22) a more rigid test. Although the curve of the solution obtained waves back and forth across that of the exact solution (see Fig. 2), as might be inferred from the integral equation

$$0 = \int_a^b G(x_k, \xi)\epsilon(\xi)\, d\xi, \qquad k = -2, -1, 0, 1, 2, \tag{27}$$

for the error $\epsilon(x)$, where the points x_k are those at which the integral equation is satisfied exactly; nevertheless the result is entirely unsatisfactory.

In locating the difficulty the first thing that was done after checking all numerical work was to obtain the five point Lagrange interpolation polynomial for the exact solution over the quadrant AB just to make sure that a fourth degree polynomial is capable of giving a good approximation. That a good approximation was obtained is seen in Table 2, and may be inferred from a glance at the curve of the exact solution; hence a situation had arisen which is the exact opposite of that occurring in Problems 1a, 2a, and 3a, wherein the solution obtained by solving the integral equation with the original technique is as good as or better than the Lagrange polynomial for the exact solution.

To check the integration scheme, values of the exact solution were substituted in the linear equations (those of (9) corresponding to $k = 1, 3, 5, 7$), and the two sides of each equation compared; thus

k	Left side of (9).	Right side of (9).
1	$14.479q$	$14.580q$
3	$12.588q$	$12.588q$
5	$8.480q$	$8.509q$
7	$4.239q$	$4.197q$

We thus see that these equations are satisfied to two digits by the charge distribution given by the Lagrange polynomial approximation of the exact solution. The fact that two widely different sets of y's almost satisfy the same set of linear equations indicates a "softness" in those equations similar to that found in the force-deflection equations of a framework near instability, or the current-voltage equations of an A. C. network near resonance. The determinant is on the verge of vanishing.[13]

Subtracting the Lagrange polynomial from that obtained by (22), we obtain a polynomial charge distribution of considerable magnitude the potential due to which or any fraction of which almost vanishes at all five points, including $x = \frac{\pi}{2} R$. This fact is peculiar from a physical standpoint, and is perhaps associated with the fact that the charge is distributed around a closed ring.

[13] Since this "softness" is in itself exceptional, it might be expected that any additional restriction would be sufficient to do away with it. With this in mind $y(x)$ was determined using the least square technique with only points $k = 1, 3, 4, 5, 7$ of the k index—just one more point than was used with the original technique. The corresponding weighting factors were $D_1 = 1.0$, $D_3 = 1.5$, $D_4 = 1.0$, $D_5 = 1.5$, $D_7 = 2.0$, these being chosen as described in the first part of Section 5. The results are given in Table 2, from which we see that the one extra point was enough to do away with almost the entire error. Taking the difference between the number of k points and the number of parameters, or y's, as a measure of the strength of the least square process, we thus have a case where a weak least square procedure gives good results, whereas the original technique fails completely.

With a given number of parameters, the greater the number of k points the better in so far as accuracy is concerned, the limitation being the amount of work involved.

With a given number of k points the question arises as to what number of parameters (degree of $y(x)$) should be used for greatest accuracy. As more y's are used the polynomial becomes more flexible, but the least square process is weakened. Enough must be used so that a polynomial of corresponding degree is capable of approximating the unknown function; however, present indications are that the accuracy is decreased and the amount of work increased by appreciably exceeding this number, because of the resulting weakening of the least square process.

It should be noted that the justification of the original technique as given in Part E of Section 4a hinges upon the smoothness of $\Phi(x)$. This justification evidently fades if the solution obtained is too wavy, which fact gives us some warning in the case of poor solutions of the type just considered. The least square technique was developed specifically to avoid such difficulties, although many problems, for example beam stability and vibration problems, have shown no necessity for refining the original technique.

Massachusetts Institute of Technology.

Key to Tables[4]

Each of Tables 1, 2, 3, and 4 contains data corresponding to a particular field; thus

Table 1. Uniform field of intensity E (originally). See Problem 1, page 318.

Table 2. Field due to two parallel oppositely charged wires on opposite sides of the cylinder. See Problem 2, page 318.

Table 3. Field corresponding to $\phi(x) = L \cos \frac{3x}{R}$. See page 324.

Table 4. Field corresponding to $\phi(x) = L \cos \frac{5x}{R}$. See page 324.

In each of these tables the exact solution is $\sigma(x)$; and the procedure by which a polynomial approximation $y(x)$ or a discrepancy $\Phi(x)$ is obtained is indicated by one of the following subscripts.

A. Original technique using equations of (9) corresponding to the four points $k = 1, 3, 5, 7$. Solution given by (22).

B. Least square technique using all eight of the equations (9), and using weighting factors (20), corresponding to the trapezoidal rule. Solution given by (23).

C. Least square technique using all eight of the equations (9), and using weighting factors (10), corresponding to Simpson's rule. Solution given by (13).

D. Original technique using equations of (9) corresponding to the four points $k = 2, 4, 6, 8$.

E. Least square technique using only the five equations of (9) corresponding to $k = 1, 3, 4, 5, 7$.[13]

P. Lagrange polynomial for the exact solution $\sigma(x)$. (Agrees with $\sigma(x)$ at points $k = 1, 3, 5, 7, 9$.)

TABLE 1

For key see page 330.

$\frac{2x}{\pi R}$	λ	$\frac{\sigma}{E}$	$\frac{y_A}{E}$	$\frac{y_B}{E}$	$\frac{y_C}{E}$	$\frac{y_D}{E}$	$\frac{y_E}{E}$	$\frac{\phi}{RE}$	$\frac{10^5\Phi_A}{RE}$	$\frac{10^5\Phi_B}{RE}$	$\frac{10^5\Phi_C}{RE}$	$\frac{10^5\Phi_D}{RE}$	$\frac{10^5\Phi_E}{RE}$
0	−2.00	.15915	.15947	.159123	.159123	.15909	.15918	1.00000	—	−1	−1	1	−1
.0625	−1.75	.15839	.15839	.158363	.158364	.15835	.15836						
.125	−1.50	.15610	.15593	.156063	.156064	.15607	.15604	.98079	35	0	0	—	7
.1875	−1.25	.15230	.15209	.152256	.152257	.15227	.15222						
.25	−1.00	.14704	.14685	.146986	.146986	.14700	.14696	.92388	—	1	1	3	1
.3125	−.75	.14036	.14024	.140307	.140307	.14031	.14030						
.375	−.50	.13233	.13229	.132284	.132283	.13229	.13229	.83147	8	0	0	—	0
.4375	−.25	.12303	.12306	.122993	.122992	.12299	.12301						
.50	0	.11254	.11262	.112519	.112518	.11251	.11254	.70711	—	0	0	1	−1
.5625	.25	.10097	.10105	.100959	.100959	.10095	.10098						
.625	.50	.08842	.08847	.088421	.088421	.08842	.08843	.55557	17	0	0	—	2
.6875	.75	.07503	.07501	.075023	.075023	.07502	.07502						
.75	1.00	.06091	.06079	.060893	.060894	.06089	.06087	.38268	—	0	0	2	0
.8125	1.25	.04620	.04600	.046171	.046172	.04617	.04614						
.875	1.50	.03105	.03080	.031006	.031007	.03101	.03096	.19509	26	0	0	—	5
.9375	1.75	.01560	.01540	.015559	.015559	.01556	.01553						
1.00	2.00	0	0	0	0	0	0	0	—	—	—	—	—

TABLE 2

For key see page 330.

$\frac{2x}{\pi R}$	λ	$\frac{\sigma R}{q}$	$\frac{y_P R}{q}$	$\frac{y_A R}{q}$	$\frac{y_B R}{q}$	$\frac{y_C R}{q}$	$\frac{y_D R}{q}$	$\frac{y_E R}{q}$	$\frac{\phi}{q}$	$\frac{10\Phi_A}{q}$	$\frac{10^2\Phi_B}{q}$	$\frac{10^2\Phi_C}{q}$	$\frac{10^2\Phi_D}{q}$	$\frac{10^2\Phi_E}{q}$
0	−2.00	.42441	.42441	.19573	.42010	.42062	.45567	.40950	2.19722	—	6.87	6.56	−2.12	.50
.0625	−1.75	.41528	.40972	.39435	.41135	.41145	.42301	.41157						
.125	−1.50	.38988	.38355	.47143	.38804	.38792	.38581	.39321	2.11460	−2.27	.51	.67	—	−1.25
.1875	−1.25	.35322	.34961	.46417	.35475	.35456	.34607	.36083						
.25	−1.00	.31111	.31111	.40451	.31546	.31529	.30552	.32000	1.89700	—	−6.22	−6.48	−2.12	−.54
.3125	−.75	.26831	.27079	.31908	.27355	.27346	.26560	.27537						
.375	−.50	.22786	.23089	.22922	.23180	.23180	.22747	.23077	1.60408	−.56	−1.11	−1.20	—	−.22
.4375	−.25	.19124	.19316	.15099	.19236	.19245	.19201	.18913						
.50	0	.15888	.15888	.09515	.15680	.15694	.15983	.15251	1.28231	—	3.58	3.34	−.27	.54
.5625	.25	.13056	.12883	.06717	.12608	.12624	.13125	.12209						
.625	.50	.10578	.10329	.06723	.10055	.10069	.10631	.09821	.95554	−1.08	2.77	2.68	—	−.19
.6875	.75	.08397	.08208	.09023	.07996	.08004	.08476	.08030						
.75	1.00	.06452	.06452	.12576	.06346	.06346	.06610	.06694	.63258	—	−2.09	−2.41	−2.22	−.18
.8125	1.25	.04688	.04943	.15813	.04958	.04949	.04951	.05584						
.875	1.50	.03055	.03515	.16636	.03625	.03612	.03392	.04382	.31472	−1.64	−2.74	−2.62	—	−1.15
.9375	1.75	.01507	.01956	.12417	.02082	.02070	.01796	.02685						
1.00	2.00	0	0	0	0	0	0	0	0	—	—	—	—	—

TABLE 3

For key see page 330.

$\frac{2x}{\pi R}$	λ	$\frac{\sigma R}{L}$	$\frac{y_A R}{L}$	$\frac{y_B R}{L}$	$\frac{y_C R}{L}$	$\frac{y_D R}{L}$	$\frac{y_E R}{L}$	$\frac{\phi}{L}$	$\frac{10\Phi_A}{L}$	$\frac{10^2\Phi_B}{L}$	$\frac{10^2\Phi_C}{L}$	$\frac{10^2\Phi_D}{L}$	$\frac{10^2\Phi_E}{L}$
0	−2.00	.47746	.69357	.49136	.49126	.45311	.51987	1.00000	—	−7.05	−6.97	2.35	−.41
.0625	−1.75	.45691	.48686	.47259	.47283	.46061	.47287						
.125	−1.50	.39700	.32622	.40252	.40282	.40549	.38977	.83147	2.06	−.52	−.70	—	3.18
.1875	−1.25	.30290	.19741	.29671	.29689	.30629	.28138						
.25	−1.00	.18272	.08899	.16919	.16917	.17966	.15766	.38268	—	7.18	7.93	2.52	.44
.3125	−.75	.04680	−.00771	.03250	.03226	.04037	.02780						
.375	−.50	−.09315	−.09859	−.10232	−.10274	−.09871	−.09985	−.19509	.46	−1.37	−.36	—	.18
.4375	−.25	−.22508	−.18678	−.22577	−.22629	−.22661	−.21777						
.50	0	−.33762	−.27261	−.32982	−.33034	−.33425	−.31922	−.70711	—	−.86	.11	.82	−.44
.5625	.25	−.42109	−.35366	−.40798	−.40839	−.41442	−.39829						
.625	.50	−.46829	−.42472	−.45523	−.45545	−.46183	−.44989	−.98079	.96	−3.54	−3.29	—	1.02
.6875	.75	−.47517	−.47782	−.46810	−.46807	−.47308	−.46975						
.75	1.00	−.44112	−.50218	−.44460	−.44429	−.44664	−.45439	−.92388	—	.81	.80	2.22	.14
.8125	1.25	−.36909	−.48427	−.38426	−.38372	−.38290	−.40116						
.875	1.50	−.26527	−.40780	−.28810	−.28747	−.28412	−.30823	−.55557	1.51	4.10	3.61	—	2.77
.9375	1.75	−.13860	−.25366	−.15866	−.15818	−.15447	−.17458						
1.00	2.00	0	0	0	0	0	0	0	—	—	—	—	—

TABLE 4

For key see page 330.

$\frac{2x}{\pi R}$	λ	$\frac{\sigma R}{L}$	$\frac{y_A R}{L}$	$\frac{y_B R}{L}$	$\frac{y_C R}{L}$	$\frac{y_D R}{L}$	$\frac{y_E R}{L}$	$\frac{\phi}{L}$	$\frac{\Phi_A}{L}$	$\frac{10\Phi_B}{L}$	$\frac{10\Phi_C}{L}$	$\frac{10\Phi_D}{L}$	$\frac{10\Phi_E}{L}$
0	−2.00	.79577	−1.93145	.77579	.76749	.68154	.31696	1.00000	—	1.07	1.15	1.85	.53
.0625	−1.75	.70181	.44541	.63240	.62549	.58918	.62652						
.125	−1.50	.44211	1.38762	.36064	.35647	.35272	.56493	.55557	−2.79	−.13	−.09	—	−5.40
.1875	−1.25	.07800	1.37530	.04107	.04019	.05547	.28842						
.25	−1.00	−.30453	.81560	−.26127	−.25892	−.23494	−.07330	−.38268	—	−.89	−.94	−.75	−.57
.3125	−.75	−.61514	.04269	−.49678	−.49180	−.46660	−.41698						
.375	−.50	−.78048	−.68222	−.63137	−.62470	−.60327	−.66587	−.98079	−.59	.42	.29	—	−.23
.4375	−.25	−.76151	−1.17000	−.64646	−.63921	−.62444	−.76973						
.50	0	−.56270	−1.30809	−.53890	−.53228	−.52527	−.70479	−.70711	—	.11	−.01	−.13	.57
.5625	.25	−.23100	−1.05148	−.32109	−.31619	−.31663	−.47379						
.625	.50	.15525	−.43176	−.02086	−.01856	−.02507	−.10595	.19509	−1.33	.08	.04	—	−2.16
.6875	.75	.50483	.44743	.31844	.31766	.30713	.34301						
.75	1.00	.73520	1.40948	.63798	.63415	.62202	.79088	.92388	—	−.11	−.08	.37	−.19
.8125	1.25	.79194	2.20480	.86345	.85728	.84596	1.12895						
.875	1.50	.66166	2.51084	.90505	.89807	.88958	1.22204	.83147	−1.98	−.08	−.03	—	−3.82
.9375	1.75	.37513	1.93207	.65749	.65220	.64784	.90845						
1.00	2.00	0	0	0	0	0	0	0	—	—	—	—	—

TABLE 5

Exact values of $K_i(\lambda)$ for $n = 1$ (three points)

λ	K_{-1}	K_0	K_1	
.25	−.09375	.93750	.15625	−.25
.75	−.09375	.43750	.65625	−.75
	K_1	K_0	K_{-1}	λ

TABLE 6

Exact values of $K_i(\lambda)$ for $n = 2$ (five points)

λ	.25	.75	1.25	1.75	
K_{-2}	.01708 98437 5	.01708 98437 5	−.02197 26562 5	−.03759 76562 5	K_2
K_{-1}	−.12304 68750 0	−.10742 18750 0	.12695 31250 0	.20507 81250 0	K_1
K_0	.92285 15625 0	.37597 65625 0	−.34277 34375 0	−.48339 84375 0	K_0
K_1	.20507 81250 0	.75195 31250 0	1.14257 81250 0	.75195 31250 0	K_{-1}
K_2	−.02197 26562 5	−.03759 76562 5	.09521 48437 5	.56396 48437 5	K_{-2}
	−.25	−.75	−1.25	−1.75	λ

TABLE 7

Exact values of $K_i(\lambda)$ for $n = 3$ (seven points)

λ	.25	.75	1.25	
K_{-3}	−.00352 47802 73437 5	−.00352 47802 73437 5	.00416 56494 14062 5	K_3
K_{-2}	.03054 80957 03125 0	.02883 91113 28125 0	−.03268 43261 71875 0	K_2
K_{-1}	−.13746 64306 64062 5	−.11329 65087 89062 5	.11802 67333 98437 5	K_1
K_0	.91644 28710 93750 0	.35247 80273 43750 0	−.28326 41601 56250 0	K_0
K_1	.22911 07177 73437 5	.79307 55615 23437 5	1.06224 06005 85937 5	K_{-1}
K_2	−.03927 61230 46875 0	−.06344 60449 21875 0	.14163 20800 78125 0	K_{-2}
K_3	.00416 56494 14062 5	.00587 46337 89062 5	−.01011 65771 48437 5	K_{-3}
	−.25	−.75	−1.25	λ

λ	1.75	2.25	2.75	
K_{-3}	.00587 46337 89062 5	−.01011 65771 48437 5	−.02232 36083 98437 5	K_3
K_{-2}	−.04464 72167 96875 0	.07498 16894 53125 0	.16213 98925 78125 0	K_2
K_{-1}	.15220 64208 98437 5	−.24513 24462 89062 5	−.51344 29931 64062 5	K_1
K_0	−.31890 86914 06250 0	.47210 69335 93750 0	.93353 27148 43750 0	K_0
K_1	.55809 02099 60937 5	−.63734 43603 51562 5	−1.10023 49853 51562 5	K_{-1}
K_2	.66970 82519 53125 0	1.27468 87207 03125 0	1.02688 59863 28125 0	K_{-2}
K_3	−.02232 36083 98437 5	.07081 60400 39062 5	.51344 29931 64062 5	K_{-3}
	−1.75	−2.25	−2.75	λ

CORRECTIONS IN "TABLES OF COMPLETE ELLIPTIC INTEGRALS"

BY CARL HEUMAN

Page 127, line 9 from below.	For page 000	read page 158
Page 130, line 14.	For or $G_0(\alpha)$	read of $G_0(\alpha)$
Page 138, line 18 from below.	For page 187	read page 177
Page 138, line 13 from below.	For page T33	read page 177
Page 139, line 14.	For page 137	read page 193
Page 139, line 7 from below.	For page 187	read page 177
Page 151 for $\alpha = 67.5°$	For 27984	read 27948
Page 151 for $\alpha = 76.8°$	For 38369	read 38639
Page 185 for $\alpha = 71°$, $\beta = 15°$	For 94145	read 84145
Page 196 for $\alpha = 0.3°$, $\beta = 87°$	For 8523	read 8623
Page 202, line 4.	For evolution	read revolution

MATHEMATICAL THEORY OF THE DIFFERENTIAL ANALYZER

By Claude E. Shannon

Introduction

The Differential Analyzer is a machine developed at the Massachusetts Institute of Technology under the direction of Dr. V. Bush for the purpose of obtaining numerical solutions to ordinary differential equations. The fundamental principles underlying the Differential Analyzer were first conceived by Lord Kelvin, but at the time it was impossible, due to mechanical difficulties, to construct a machine of the type he contemplated. The same principles were discovered independently by Dr. Bush and his associates, and the first Differential Analyzer was completed in 1931. The inherent mechanical difficulties were overcome by means of several ingenious devices such as the torque amplifiers, and backlash correcting units, and through improved machine working technique. Since that time, several other machines have been built in various parts of the world, and have been used in solving many problems arising in engineering, physics and other branches of science.

Detailed descriptions of the mechanical operation of the machine, and methods of setting up problems have been given in several papers (1, 3). For our purposes we may briefly summarize the mode of operation as follows. Each term or variable in the given equation or equations is represented on the machine by a certain shaft. The value of the variable at any instant is proportional to the number of revolutions of the corresponding shaft from some fixed position. These shafts are interconnected by means of mechanical units which enforce certain mathematical relations between the number of turns of the interconnected shafts. The most important mechanical units are of four types;[1] gear boxes, adders, integrators and input tables. With these units, each

[1] Some other special units have been developed. The Differential Analyzer at M.I.T. has a multiplying table for obtaining the product of two terms, and the one at Manchester has a unit useful in solving mixed difference differential equations. These, however, are much less important than those mentioned above; the multiplying table, in fact, can always be replaced by two integrators and an adder as will be shown later.

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relation implied by the given equations is forced to hold between the corresponding shafts. For example, if in the differential equation the sum of two terms is equal to a third term, the corresponding shafts on the Analyzer are connected by an adding unit. If the terms x, y and dy/dx appear in the equation, the corresponding shafts would be interconnected through an integrator to enforce the relation $y = \int dy/dx\, dx$, etc. When the shaft representing the independent variable is turned, all other shafts are therefore constrained to turn in accordance with the equation. Thus, a numerical solution may be obtained by counting the number of turns of the dependent variable shaft corresponding to equal increments of the independent variable and plotting the result as a curve, or the machine may be set up to plot its own curves automatically by means of a special output table.

When the Differential Analyzer was first built it was thought that all functional relationships between terms of the equation being solved would have to be introduced into the machine by means of input tables. However, in connection with a problem in ballistics in which the function x^2 was desired, it was noted that by connecting an integrator to perform the operation $2\int_0^x x\, dx$, the function could be obtained without an input table. Soon it was found that practically all the important simple functions could be "generated" using only integrators and adders. This is done by setting up an auxiliary equation on the Analyzer whose solution is the desired function. In a thesis written in 1932, J. Guerrieri (2) describes interconnections for generating most of the elementary functions.

In this paper the mathematical aspects of the Differential Analyzer will be considered. The most important results deal with conditions under which functions of one or more variables can be generated, and conditions under which ordinary differential equations can be solved. Some attention will also be given to approximation of functions (which cannot be generated exactly), approximation of gear ratios, and automatic speed control.

We shall assume throughout that all ordinary differential equations considered have unique solutions and that formal processes of differentiation, integration, etc. are valid in the region of interest. For total differential equations it is not necessary that the equations be integrable but we assume that a solution exists along any curve in the region. The reason for this will appear later.

The Differential Analyzer will be idealized by assuming that we have available an unlimited number of each of the following perfect units.

1. *Integrators.* Given two shafts u and v, the integrator constrains a third shaft w to rotate in accordance with $w = \int_{v_0}^{v} (u + a)\, dv$, where a is an arbitrary constant, for all variations of u and v. In actual integrators the maximum value of $| u + a |$ is limited, but by changing scale factors it can be made as great as desired so that except for poles of u the integration can be performed. The constant a is the initial setting of the integrator.

2. *Adders.* Given two shafts u and v, an adder constrains a third shaft w to turn as $(u + v)$, for all variations of u and v. Except for backlash and tooth ripple, the actual differential adder units fulfill these conditions. It is easily seen that any interconnection of adders with free shafts $X_1 \cdots X_n$, Y, will enforce a relation $Y = \sum_{k=1}^{n} a_k X_k$ where the a's are real constants, and for convenience we will speak of such an interconnection as an adder. By connecting simple adders in series we can make all the a's unity and get $Y = \sum_{k=1}^{n} X_k$.

3. *Input Tables or Function Units.* Given a shaft x, an input table constrains a second shaft y to rotate as $y = f(x)$, where $f(x)$ is an arbitrary given function with only a finite number of finite discontinuities.

4. *Gear Boxes.* Given a shaft x, a gear box of ratio k forces a second shaft to rotate as kx. Since by making the integrand u identically zero in the integrator we get $w = av$, the gear boxes are theoretically superfluous, being used for purposes of economy. Therefore, we will not consider gear boxes, except to show that any ratio k may be approximated using only two sizes of gear pairs.

Finally, we assume that each of the above elements is capable of performing its particular constraint if and only if there is not more than one source of drive to each shaft. By a source of drive is meant any one of the following: the independent variable shaft, the w shaft of an integrator, the w shaft of an adder (or Y in the general sense), the $f(x)$ shaft of an input table, or the kx shaft of a gear box. This condition is extremely important in making Differential Analyzer setups. It places a restriction on possible interconnections of units, and forms the basis of the present analysis.

We shall say that we can solve a system of ordinary differential equa-

tions with independent variable x and dependent variables $y_1 \cdots y_n$ if and only if a setup can be found using the above elements and satisfying the source of drive assumption, such that when the independent variable shaft x is turned, shafts $y_1 \cdots y_n$ are constrained to turn in accordance with the equations for any given initial conditions. We shall say a system of total differential equations can be solved if a set of connections can be found such that when a set of independent variable shafts $x_1 \cdots x_m$ are turned in any manner whatever, dependent variable shafts $y_1 \cdots y_n$ are constrained to turn in accordance with the equations for arbitrary given initial conditions. In most of the theorems we shall consider setups containing only integrators and adders.

Fundamental Solvability Condition

THEOREM I. A necessary and sufficient condition that a system of ordinary differential equations can be solved using only integrators and adders is that they can be written in the form

$$\frac{dy_k}{dy_1} = \sum_{i,j=0}^{n} a_{ijk} y_i \frac{dy_j}{dy_1} \qquad k = 2, 3 \cdots n \qquad (1)$$

where $y_0 = 1$ (introduced to make notation compact), y_1 is the independent variable, and $y_2 \cdots y_n$ are dependent variables, among which are the dependent variables of the original system.

Proof. The condition is necessary. Suppose the original system can be solved using only integrators and adders. The dependent variables must appear either as the output of adders or integrators. They may all be considered as the outputs of integrators by making those from adders the variables of integration of integrators with constant unit integrand. Now let there be $(n - 1)$ integrators and let the outputs be labeled $y_2 \cdots y_n$. Each displacement (integrand) must be driven by one of three possible sources: the independent variable y_1, the output of an integrator, or the output of an adder (general sense). An adder must be driven by the sum of certain of the y's including possibly y_1. These are all special cases of a source of drive of the form $\sum_{i=1}^{N} b_{ik} y_i$ for the kth integrator, where the b's are constants. The b's can obviously represent gear ratios or complex adder interconnections, or without loss of generality we may take them equal to 0 or 1. The kth integrator may have an initial displacement in addition to its source of drive value. Let this be b_{0k}. Then the kth integrator will have an integrand $\sum_{i=0}^{n} b_{ik} y_i$ where for convenience we introduce $y_0 = 1$. By exactly the

same argument the variables of integration of the integrators will be of the form $\sum_{j=1}^{n} c_{jk} y_j$. The integrators place the following constraints on the system:

$$y_k = \int \sum_i b_{ik} y_i \, d \sum_j c_{jk} y_j \qquad k = 2, 3 \cdots n \qquad (2)$$

$$= \int \sum_i b_{ik} y_i \sum_j c_{jk} \, dy_j \qquad k = 2, 3 \cdots n \qquad (3)$$

$$= \int \sum_{i,j} a_{ijk} y_i \, dy_j \qquad k = 2, 3 \cdots n \qquad (4)$$

where $a_{ijk} = b_{ik} c_{jk}$. Now differentiating both sides with respect to y_1:

$$\frac{dy_k}{dy_1} = \sum_{i,j} a_{ijk} y_i \frac{dy_j}{dy_1} \qquad k = 2, 3 \cdots n \qquad (1)$$

These are the equations (1).

The condition is sufficient. This is true since the system (1) can be integrated to the form (4) and these represent a setup using only integrators and adders with not more than one source of drive to each shaft.

Generation of Functions

We will say that a function of a single variable $y = f(x)$ can be generated if there is a setup using only integrators and adders such that a shaft may be turned independently as x and a second shaft is forced to turn as y. It follows from Theorem I that if $f(x)$ can be generated there must exist a set of equations (1) such that if $y_1 = x$, then y_2 (say) is equal to $f(x)$. A function of n variables $F(x_1 \cdots x_n)$ can be generated if there is a setup such that n shafts $x_1 \cdots x_n$ can be turned independently and another will turn as F.

Functions of a single variable have been classified in the following way. If there is a relation of the form:

$$y = a_0 + a_1 x + a_2 x^2 + \cdots a_m x^m \qquad (5)$$

where m is a positive integer and the a's are real constants then y is called a polynomial in x, or a rational integral function of x. If

$$y = \frac{F_1(x)}{F_2(x)} \qquad (1)$$

where F_1 and F_2 are polynomials in x, then we say y is a rational function of x. If we have:

$$R_0(x) + yR_1(x) + y^2 R_2(x) + \cdots + y^n R_n(x) = 0 \qquad (7)$$

where $R_1 \cdots R_n$ are rational functions of x, then y is said to be an algebraic function of x. If there is no relation of this form then y is called a transcendental function of x. Transcendental functions may be divided into two classes. If there is a differential equation of the form:

$$\sum_i A_i x^{n_i} y^{n_{0i}} (y')^{n_{1i}} (y'')^{n_{2i}} \cdots (y^{(m)})^{n_{mi}} = 0 \tag{8}$$

where $y^{(k)} = d^k y/dx^k$, the A's are constants, and the powers are integral, (i.e., if a polynomial in $x, y, y' \cdots y^{(m)}$ vanishes identically) then y is said to be an algebraic transcendental function of x. If no relation of this type exists the function is called hypertranscendental, or transcendentally-transcendental. Obviously all algebraic functions are non-hypertranscendental, since by multiplying (7) by all denominators and differentiating we obtain an expression of the form (8). In fact only a very few of the common analytic functions are hypertranscendental, the best known being the Gamma function and Riemann's Zeta function. $\left(\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx;\ \xi(s) = \sum_{k=0}^{\infty} \frac{1}{k^s}\right)$. The first was shown to be hypertranscendental by Holden, and the last by Hilbert. The classification of some of the common functions is given in Table I.

TABLE I

Functions of One Variable

Transcendental		Algebraic		
Hypertranscendental	Algebraic-transcendental	Irrational Algebraic	Rational	
			Quotients of polynomials.	Integral a, x, X^2 etc. polynomials.
Gamma function. Zeta function.	Exponential and logarithmic. Trigonometric and hyperbolic and inverses. Bessel functions. Elliptic functions and integrals. Probability function.	x^m, m a rational fraction. Solutions of an algebraic equation in terms of a parameter.		

THEOREM II. A function of one variable can be generated if and only if the function is not hypertranscendental.

Proof. First we will show that if a function can be generated, it is not

hypertranscendental. Any function $f(x)$ that can be generated must satisfy a set of equations of the form (1), with $y_1 = x$ and $y_2 = f(x)$. If we differentiate (1) $(n - 2)$ times we will have a total of $(n - 1)^2$ equations, from which we may eliminate the $n^2 - 2n$ variables

$$y_3, y_3', \cdots y_3^{(n-1)}; y_4, y_4', \cdots y_4^{(n-1)}; \cdots; y_n, y_n', \cdots y_n^{(n-1)}$$

for example by Sylvester's method, involving only multiplication and addition and therefore resulting in a relation of the form (8), with $x = y_1$, $y = y_2$, and $1 = y_0$.

To prove that any non-hypertranscendental function can be generated, we will show that (8) can be written in the form (1). Let the left member of (8) be ϕ. Differentiating both sides with respect to x we obtain:

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} y' + \frac{\partial \phi}{\partial y'} y'' + \cdots + \frac{\partial \phi}{\partial y^{(m)}} y^{(m+1)} = 0$$

And except for points where $\frac{\partial \phi}{\partial y^{(m)}} = 0$ we have:

$$y^{(m+1)} = \frac{\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} y' + \cdots + \frac{\partial \phi}{\partial y^{(m-1)}} y^{(m)}}{\frac{\partial \phi}{\partial y^{(m)}}} = \frac{P_1(x, y, y' \cdots y^{(m)})}{P_2(x, y, y' \cdots y^{(m)})}$$

where P_1 and P_2 are polynomials in $x, y, y', y'' \cdots y^{(m)}$. Let $y_1 = x$, $y_2 = y$, $y_3 = y'$, $\cdots y_{m+2} = y^{(m+1)}$. Then we have

$$\frac{dy_k}{dy_1} = y_{k+1} \qquad k = 2, 3, \cdots (m + 2)$$

with the additional condition that

$$y_{(m+3)} = \frac{P_1(y_1, y_2 \cdots y_{(m+2)})}{P_2(y_1, y_2 \cdots y_{m+2})}$$

The problem now is to reduce this relation to the form of equations (1). First consider the function P_1. This is a sum of products of the variables $y_1 \cdots y_{m+2}$. (Since the powers are integral they may be considered as products of a repeated factor). Let the first term of the numerator be $u_1 u_2 \cdots u_s$, where the u's are certain of the y's. Now let

$$\frac{dy_{m+4}}{dy_1} = u_1 \frac{du_2}{dy_1} + u_2 \frac{du_1}{dy_1}$$

so that $y_{m+4} = u_1 u_2$. Next let:

$$\frac{dy_{m+5}}{dy_1} = u_3 \frac{dy_{m+4}}{dy_1} + y_{m+4} \frac{du_3}{dy_1}$$

Hence $y_{m+5} = y_{m+4} \cdot u_3 = u_1 u_2 u_3$. Continuing in this manner we finally get $y_{m+s+2} = u_1 u_2 \cdots u_s$. Each term in the numerator and denominator is treated in exactly the same way, continuing the series of equations until there is a y for each product of P_1 and P_2. Let the y's corresponding to the terms of P_1 be $v_1, v_2 \cdots v_r$ and those corresponding to the terms of P_2 be $w_1, w_2 \cdots w_t$. Then we have reduced the equation (12) to the condition that:

$$y_{m+3} = \sum_1^r v_k / \sum_1^t w_k .$$

Our final step is to reduce this relation to the form of the equations (1). Suppose the last y, w_t, was y_{g-1}. Let:

$$\frac{dy_g}{dy_1} = - y_{g+1} \frac{d \sum w}{dy_1}$$

$$\frac{dy_{g+1}}{dy_1} = 2y_g \frac{dy_g}{dy_1}$$

Consequently $y_g = 1/\sum w$. Now by making:

$$\frac{dy_{m+3}}{dy_1} = y_g \frac{d \sum v}{dy_1} + \sum v \frac{dy_g}{dy_1}$$

we get $y_{m+3} = \sum v / \sum w$ and have reduced (8) to the form (1).

THEOREM III. If a function of one variable $y = f(x)$ can be generated, then its derivative $z = f'(x)$, its integral $w = \int_a^x f(x)\,dx$, and its inverse $x = f^{-1}(y)$ can be generated.

Proof. To prove the first part, take the derivative of (8). This gives two equations between which we may eliminate y with processes involving only multiplication and addition, thus resulting in a relation of the form (8) with y missing. Replacing y' by z, y'' by z' etc., shows that z is non-hypertranscendental if y is.

To show the second part, merely replace y by w', y' by w'' etc. in (8), thus obtaining another relation of this form. Therefore if y is non-hypertranscendental, then w is.

For the third part replace y' by $1/x'$ (where $x' = dx/dy$)y'' by

$-x''/(x')^3$ etc. Since all the derivatives of y may be expressed as quotients of polynomials in the derivatives of x, we may reduce the resulting expression to the form (8) with x and y interchanged by multiplying by all denominators. That is, the inverse of a non-hypertranscendental function is non-hypertranscendental.

THEOREM IV. If two functions f and g can both be generated, then the functional product $y = f(g(x))$ can be generated.

Proof. This will be shown by writing two systems of equations, each of the form (1) as a single larger system of the same form. Suppose g satisfies a set of this type with $y_1 = x$ and $y_2 = g$, k running from 2 to n. Now f also satisfies a system of this type and since the argument of f is g we replace y_1 in this system by y_2 and let the y subscripts in the f system run from $n + 1$ to $n + m$ (m being the number of equations in the f system.) Thus we have a group of equations of the form (1) with $y_1 = x$ and $y_{m+2} = f(g(x))$.

Although, as Theorem II shows, the only functions which can be generated exactly are non-hypertranscendental, we can approximate a much broader class of functions, using only integrators.

THEOREM V. Any function $f(x)$, which is continuous in a closed interval $a \leqq x \leqq b$, can be generated in this interval to within any prescribed allowable error $\epsilon > 0$ using only a finite number of integrators. That is, a setup can be found generating a function $F(x)$ such that

$$| F(x) - f(x) | < \epsilon$$

for $a \leqq x \leqq b$.

Proof. We base our demonstration on Weierstrass' famous theorem which states that any such function $f(x)$ can be approximated by a polynomial $F(x)$ of degree n

$$F(x) = \sum_{k=0}^{n} a_k x^k$$

by taking n sufficiently large. Now let:

$$\frac{dy_j}{dy_1} = a_j j! + y_{j+1} \qquad j = 2, 3 \cdots n$$

$$y_{m+1} = 0$$

Then this system, of the form (1), satisfies (19) if we let $y_1 = x$ and $y_2 = F(x)$. Moreover, this setup requires only integrators, the additive constants being merely initial integrator settings. Hence the theorem. If we allow stopping the machine and turning shafts ahead by hand, we

can obviously broaden the conditions of Theorem V to all functions continuous except for a finite number of finite discontinuities.

We now proceed to generalize some of these concepts and theorems to functions of more than one variable.

THEOREM VI. A function of m variables $y_{m+1} = f(y_1 \cdots y_m)$ can be generated if and only if it satisfies a set of total differential equations of the form:

$$dy_k = \sum_{i,j=0}^{n} a_{ijk} y_i dy_j \qquad k = (m+1), (m+2) \cdots n \qquad (10)$$

where $y_0 = 1$ and the a's are real constants.

The proof follows exactly the same plan as was used in proving Theorem I, both for necessity and sufficiency. A solution of equations (1) has been shown to be a non-hypertranscendental function of one variable. Now (21) can be considered as a generalization of (1), and we will say that a function of m variables satisfying a set of equations (1) is a non-hypertranscendental function of these m variables. With this definition we have, as a generalization of Theorem II, the proposition that a function of m variables can be generated if and only if it is not hypertranscendental in these variables. Obviously a necessary condition for a function of m variables to be non-hypertranscendental is that it be non-hypertranscendental in each single variable when all the others are replaced by arbitrary constants. Thus $x + \Gamma(y)$ is a hypertranscendental function of x and y since replacing x by 0 gives the hypertranscendental function of one variable $\Gamma(y)$. Some functions of more than one variable that are not-hypertranscendental are $x + y$, $x \cdot y$, x^y, $\log_x y$, and combinations of these and non-hypertranscendental functions of one variable.

As a generalization of Theorem IV we state the proposition:

THEOREM VII. If two functions of several variables, $f(x_1 \cdots x_n)$ and $g(y_1 \cdots y_m)$ can both be generated, then it is possible to generate any functional product, for example $\phi(x_2, x_3 \cdots x_n, y_1, y_2 \cdots y_m) = f(g, x_2, \cdots x_n)$.

This may be proved by the same method as Theorem IV, combining the two systems of equations into a single larger system of the type (10).

Theorem V may also be generalized for functions of more than one variable, but integrators alone are no longer sufficient:

THEOREM VIII. Given any function of n variables $f(x_1 \cdots x_n)$, continuous in all variables in a closed region of n-space $a_k \leqq x_k \leqq b_k$, $k = 1, 2 \cdots n$, we can generate a function $F(x_1 \cdots x_n)$ using only a finite

number of integrators and adders such that within the region $a_k \leqq x_k \leqq b_k$

$$|f - F| < \epsilon$$

where ϵ is an arbitrarily small prescribed positive number.

Proof. A generalization of Weierstrass' theorem states that we may approximate f in the manner described with a polynomial F in the variables $x_1, \cdots x_n$. Since a polynomial is a non-hypertranscendental function according to our definition, it can be generated with integrators and adders and hence we have the theorem.

The first part of Theorem III can be generalized for functions of more than one variable as follows.

THEOREM IX. If a function of m variables $f(x_1 \cdots x_n)$ can be generated its partial derivative with respect to any one variable, say x_1 can be generated.

Proof. From theorem VI, if a function can be generated it satisfies a set of equations of the form

$$dy_i = \sum_{j=1}^{m} A_{ij} dx_i + \sum_{j=1}^{m} B_{ij} dy_j \qquad i = 1, 2 \cdots s$$

where $x_1 \cdots x_m$ are the independent variables, $y_1 \cdots y_s$ are dependent variables, the A's and B's are linear forms in the variables, and y_1 (say) is equal to f. Dividing these equations by dx_1 and setting $0 = dx_2 = dx_3 = \cdots = dx_m$ we have:

$$\frac{\partial y_i}{\partial x_1} = A_{i1} + \sum_{j=1}^{s} B_{ij} \frac{\partial y_j}{\partial x_1} \qquad i = 1, 2 \cdots s$$

or

$$\sum_{j=1}^{s} (B_{ij} - \delta_{ij}) \frac{\partial y_j}{\partial x_1} = A_{i1} \qquad i = 1, 2 \cdots s.$$

Solving for $\frac{\partial y_1}{\partial x_1}$ by Cramer's rule:

$$\frac{\partial y_1}{\partial x_1} = \frac{\begin{vmatrix} A_{11} & B_{12} \cdots B_{1s} \\ A_{21} & B_{22} \cdots B_{2s} \\ \cdots\cdots\cdots\cdots \\ A_{s1} & B_{s2} \quad B_{ss} \end{vmatrix}}{|B_{ij} - \delta_{ij}|} = \frac{P_1}{P_2}$$

where P_1 and P_2 are polynomials in the variables. This equation may be reduced to the form of equations (10) by exactly the same method as was used in reducing equation (9) which was of the same form.

The last part of Theorem III can also be generalized as follows:

THEOREM X. If a function of n variables $y = f(x_1 \cdots x_n)$ can be generated, its inverse with respect to any one variable $x_1 = F(y, x_2 \cdots x_n)$ can be generated.

Proof. Taking the total differential of y we have:

$$dy = \sum_{i=1}^{n} f'_{x_i}\, dx_i .$$

Hence:

$$dx_1 = 1/f'_{x_1}\left(dy - \sum_{i=2}^{n} f'_{x_i}\, dx_i\right).$$

Or:

$$x_1 = \int 1/f'_{x_1}\, dy = \sum_{i=1}^{n} \int f'_{x_i}/f'_{x_1}\, dx_i$$

Now since f could be generated the terms f'_{x_i} $(i = 1, 2, \cdots n)$ can be generated by the preceding theorem. Reciprocals and quotients are not hypertranscendental and therefore the terms $1/f_{x_1}$ and $-f_{x_i}/f_{x_1}$ can be generated. It follows then that x_1 can be obtained by generating these integrands, integrating with respect to the proper variables and adding the results.

Systems of Equations

We are now in a position to prove the following general theorem on the differential analyzer:

THEOREM XI. The most general system of ordinary differential equations:

$$f_k(x; y_1, y_1' \cdots y_1^{(m)};\ y_2, y_2' \cdots y_2^{(m)} \cdots y_n, y_n' \cdots y_n^{(n)}) = 0 \qquad k = 1, 2, \cdots n \tag{11}$$

of the mth order in n dependent variables can be solved on a differential analyzer using only a finite number of integrators and adders providing the functions f_k are combinations of non-hypertranscendental functions of the variables.

Before proving this it is necessary to demonstrate a preliminary

lemma. The natural procedure would be to solve the equation $f_1 = 0$ for $y_1^{(m)}$, $f_2 = 0$ for $y_2^{(m)}$ etc:

$$y_k^{(m)} = \phi_k(x; y_1, y_1' \cdots y_1^{m}; y_2 y_2' \cdots y_2^{(n)} \cdots y_n y_n' \cdots y_n^{(m)}).$$

It may happen, however, that $y_1^{(m)}$ does not appear in f_m but does appear in some other of the functions. We will first show that by taking derivatives of the equations (11) and rearranging the order, an equivalent system can be found in which the highest ordered derivative of y_1 appearing in the first equation is not exceeded in any of the other equations, the highest ordered derivative of y_2 appearing in the second equation is not exceeded in any of the others, etc.

First note that if f_k be considered a function of the independent variables $(x, y_1, y_1', \cdots y_1^{(m)}, y_2, y_2', \cdots y_2^{(m)}, \cdots y_n, y_n', \cdots y_n^{m})$ then taking the derivative of $f_k = 0$ gives an equivalent equation (providing the boundary conditions are adjusted in accord with the original equation) in which the highest ordered derivative of each variable appearing has been increased by one. Also, if the original function f_k involved no hypertranscendental functions, then by Theorem IX the derived function will not, and can be generated.

For our present purposes the essential part of equations (11) is the set of values of the highest ordered derivatives of the different variables appearing in these equations. These may be tabulated in a square array as follows:

	y_1	y_2	$\cdots$	y_n
f_1	a_{11}	a_{12}	$\cdots$	a_{1n}
f_2	a_{21}	a_{22}	$\cdots$	a_{2n}
f_3	a_{31}	a_{32}	$\cdots$	a_{3n}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
f_n	a_{n1}	a_{n2}	$\cdots$	a_{nn}

where a_{jk} is the order of the highest ordered derivative of y_k appearing in function f_j. These a's are integers which may have values from 0 to m, or if the variable does not appear at all, we may give the a a special symbol λ. Taking the derivative of f_k has the effect of adding unity to each element of the kth row except for λ's which remain the same. Two rows of the array may be interchanged since this merely means renumbering the functions. We propose to show that by rearrangement and taking derivatives we can always find a new system where a_{11} is

not exceeded in the first column, a_{22} is not exceeded in the second column, and in general a_{kk} is not exceeded in the kth column.

This will be shown by mathematical induction. We will first show it to be true for $n = 2$. For two variables we have the array:

	y_1	y_2
f_1	a_{11}	a_{12}
f_2	a_{21}	a_{22}

If more than one of the letters is λ the system is degenerate. If one of the letters, say a_{11} is λ we may interchange the rows and then differentiate the second row until the new $a_{22} \geqq a_{12}$. If none of the letters are λ then either $a_{22} = a_{12}$ or one is smaller. If they are equal the rows may be interchanged if necessary to make $a_{11} \geqq a_{21}$. If one is smaller, let this row be differentiated until they are equal and proceed as before. Thus, the theorem is true for $n = 2$.

Now, assuming it true for n, we will show that it must then be true for $n + 1$, and thus complete the induction. By hypothesis then, given any $(n + 1)^2$ array, we can find an equivalent system by differentiating and rearranging the first n rows:

	y_1	y_2	$\cdots$	y_n	y_{n+1}
f_1	a_{11}	a_{12}	$\cdots$	a_{1n}	a_{1n+1}
f_2	a_{21}	a_{22}	$\cdots$	a_{2n}	a_{2n+1}
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
f_n	a_{n1}	a_{n2}	$\cdots$	a_{nn}	a_{nn+1}
f_{n+1}	a_{n+11}	a_{n+12}	$\cdots$	a_{n+1n}	$a_{n+1}a_{n+1}$

such that $a_{kk} \geqq a_{jk}$, $k, j = 1, 2 \cdots n$. We may also assume that $a_{kk} > a_{kn+1}$ for every $k \leqq n$, since, if not, we can make it so by differentiating all the first n functions simultaneously. Now two possibilities arise: (1) $a_{n+1,n+1} \geqq a_{j,n+1}$, $j = 1, 2 \cdots n$. In this case the system is satisfactory as it stands. (2) If this condition does not obtain there must be some $a_{jn+1} > a_{n+1n+1}$. Suppose this is a_{1n}. Now let the last row be differentiated until one of the following three possibilities obtains: (1) $a_{n+1n+1} = a_{1n+1}$. In this case the system is satisfactory. (2) $a_{n+1,1} = a_{11}$. In this case interchange rows 1 and n and the system is satisfactory. (3) $a_{n+1s} = a_{ss}$ for one or more values of s between 1 and $n + 1$. In this case continue differentiating both equations s and $n + 1$ until one of the following three occurs: (1) $a_{n+1,n+1} =$

a_{1n+1} or $a_{n+11} = a_{11}$. Proceed as in (1) and (2) above. (2) $a_{s1} = a_{11}$ or $a_{sn+1} = a_{1n+1}$. Interchange rows s and $n+1$ and proceed as in (1). (3) The maximum in some other column is reached by the corresponding elements in the rows being differentiated. In this case annex this row or rows to those already being differentiated and continue by the same method. It is easy to see that the process must converge after a finite number of steps since all the elements in the first and last columns except a_{11} and a_{1n+1} cannot be λ. This completes the induction.

It is now comparatively simple to prove Theorem XI. First find, by the method just described, a system equivalent to (11) in which the highest ordered derivative of y_k appearing in equation k is not exceeded in any of the other equations. Let this order be p_k. Now it has been shown that any combinations of non-hypertranscendental functions can be generated (Theorems IV and VII), and therefore the functions f_k can be generated. Hence by the theorem on inverse functions, (Theorem X) we can generate the functions

$$y_k^{(p_k)} = \psi_k(x, y_1, y_1' \cdots y_2, y_2' \cdots \cdots y_n, y_n' \cdots)$$

The variables $y_k^{(v)}$, $v < p_k$ may be obtained by integration of $y_k^{(p_k)}$ and it follows that the system may be solved using only integrators and adders.

Approximation of Factors with Gear Ratios

In setting up problems on a Differential Analyzer it is frequently necessary to introduce constant multiplying factors. This may be done in several ways. As was pointed out, an integrator set with a constant integrand k gives $w = \int_0^v k\,dv = kv$. A second method is to get a rough approximation to k by means of gears whose overall ratio is say k'. Another set of gears is used to get a rough approximation k'' to $(k - k')$ and the variable $k'x$ combined with $k''x$ through an adder to give a second order approximation to k. It is easy to show that proceeding in this way we can obtain a ratio t such that $| t - k | < \epsilon$ where ϵ is an arbitrarily small prescribed positive number, assuming only that we have an unlimited number of adders and of one size of gear pairs with ratio $a \neq 0, 1$. A third way to obtain the ratio k is by the use of gears only. We will now show that any ratio can be approximated using only a finite number of two sizes of gear pairs. More precisely we have

THEOREM XII. Given two gear ratios a and b, neither of which is zero or one, and such that b is not a rational power of a, we can find

positive or negative integers u and v such that $0 \leqq |a^u b^v - k| < \epsilon$ where ϵ is an arbitrarily small prescribed number.

Without loss of generality we take $a > 1$; for if not inverting the direction of the gears gives a ratio $1/a > 1$. It will first be shown that a ratio U may be obtained such that

$$1 < U < 1 + \delta$$

with δ arbitrarily small. To do this integers x and y must be found such that

$$1 < a^x b^y < 1 + \delta$$

Since $a > 1$, we may take logarithms to the base a leaving the inequalities in the same direction.

$$0 < x + y \log_a b < \log_a (1 + \delta) = \mu$$

Since $\log_a b$ is irrational we can satisfy this by well known theorems in diophantine approximation. Thus we can obtain the gear ratio U. If U is connected in tandem with itself r times with r so chosen that

$$U^{r-1} < k \leqq U^r$$

then the difference between U^r and k will be less than δk and is therefore arbitrarily small.

If b is a rational power of a, say $a^{m/n} = b$ where m/n is rational in its lowest terms, then a necessary and sufficient condition that a ratio be obtainable is that it be of the form $a^{k/n}$ where k is any integer. First any ratio will be of the type $a^x b^y = a^{(xn+ym)/n}$ which is of this form. The sufficiency of the condition follows from the fact that the diophantine equation $xn + ym = k$ has a solution in integers for every integer k, if n and m are relatively prime.

Automatic Speed Control

An important part of the Differential Analyzer control circuit is the automatic speed control. The integrator outputs have a maximum speed s, beyond which slippage or vibration is apt to occur. To prevent exceeding this speed, the independent variable shaft might be turned at such a speed that only when the integrators are at maximum displacement is this speed attained. However, with such a system the solution would take a longer time than necessary, for during most of the solution none of the integrators would be at maximum displacement. The automatic speed control is a device which makes the independent variable

turn at such a speed that the integrator which is turning fastest is going at the speed s. By the use of some rather broad assumptions it is possible to get a rough estimate of the time saved by this method. The results of this theorem were compared with several experimental solutions and the error was in all cases less than 7%.

THEOREM XIII. If the displacements of n integrators driven by the independent variable with maximum speed s and integrating factor b are assumed random with respect to the independent variable between limits of $-a$ and $+a$, and if the speed is limited only by these and the independent variable x with a maximum speed of r, then the expected average speed of the independent variable will be:

$$\bar{\dot{x}} = \frac{1}{[nab/x(n+1)s] + [1/r(n+1)] \cdot [(s/abr)^n]}$$

Proof:

$$\bar{\dot{x}} = \frac{\int_{x=0}^{x=u} dx}{\int_{x=0}^{x=u} dt}$$

Where u is the maximum value of x and t the time. Now since $dt = \frac{dx}{dx/dt}$ we have:

$$\bar{\dot{x}} = \frac{u}{\int_0^u dt/dx \cdot dx}$$

If we let y be the displacement of the integrator with the largest displacement, then the probability that this is between y and $y + dy$ is:

$$n(y/a)^{n-1}\, dy/a$$

The expected number of revolutions of x in this condition will then be:

$$dx = n(y/a)^{n-1} u\, dy/a$$

If $y \leqq s/br$, the speed will be limited by the independent variable, and dx/dt will be r. If $y > s/br$ the speed will be limited by the integrators and dx/dt will be x/yb. For the first case:

$$dt = un/r(y/a)^{n-1}\, dy/a.$$

For the second case:

$$dt = yb/snu(y/a)^{n-1}\, dy/a.$$

Therefore:

$$\bar{\dot{x}} = \frac{u}{\int_{y=0}^{y=\frac{s}{br}} \frac{un}{ar}\left(\frac{y}{a}\right)^{n-1} dy + \int_{y=\frac{s}{br}}^{y=a} \frac{bnu}{s}\left(\frac{y}{a}\right)^{n} dy}$$

Or

$$\bar{\dot{x}} = \frac{1}{\frac{nab}{(n+1)s} + \frac{1}{r(n+1)}\left(\frac{s}{abr}\right)^{n}}$$

COROLLARY. If the maximum speed of the independent variable be made infinite, the solution time will be reduced in the ratio of $n/(n+1)$ to what it would be if x were given a fixed speed $v = s/ab$ so that at maximum displacement the integrators just reached the maximum allowable speed.

This follows from the general expression by allowing r to approach infinity.

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TRAJECTORIES OF MONOENERGETIC ELECTRONS, IN AN ARBITRARY STATIC ELECTROMAGNETIC FIELD, IN THE NEIGHBORHOOD OF A GIVEN TRAJECTORY*

By L. A. MacColl

1. Introduction

Much of the existing theory of electron optics is concerned with the study of the trajectories in the neighborhood of the axis of a static axially symmetric electromagnetic field.[1] In the present paper this theory is generalized, in that we consider electrons moving with fixed total energy E in an arbitrary static electromagnetic field, and we study the trajectories in the neighborhood of an arbitrary trajectory T_0. Specifically, we investigate the various types of images which can be formed, the conditions for their formation, the various types of distortion to which the images are subject, etc. This general problem seems to have been discussed on only one previous occasion, and then with but little overlapping of the discussion given here.[2]

We confine our attention to "first order effects," in the sense that certain distances, which characterize the departures of the trajectories under consideration from the basic trajectory T_0, are assumed to be so small that their squares and products can be neglected. Thus our considerations have nothing in common with those parts of the ordinary optical and electron-optical theories which deal with defects of the images.

Although most of the discussion is carried on in terms of trajectories of electrons, it will be apparent that most of the theory is equally applicable to other cases of motion of particles in fields of force, to ordinary (photon-) optics, and to various problems in pure mathematics.

2. A special coordinate system associated with T_0

It is assumed that the portion of T_0 under consideration is an analytic arc, having no singular points, and having no points in common with the surface represented by the equation $E - V = 0$, where V is the

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[1] See, for instance, Johannes Picht, Einführung in die Theorie der Elektronenoptik, Leipzig, 1939.

[2] M. Cotte, Annales de physique, series 11, tome 10 (1938), pp. 333–405.

potential energy function of the field.[3] It is also assumed that T_0 has a uniquely defined principal normal at every point of the arc. The theory can be extended to such cases, in which these assumptions are not all satisfied, as are of physical importance by means of fairly obvious supplementary conventions.

Let O be an arbitrarily chosen fixed point on T_0.

Consider a point P, which may be on T_0 or not. If P is not too remote from T_0, there is associated with P a unique point Q on T_0, such that the distance PQ is an absolute minimum. We confine our attention to points P such that the associated points Q are uniquely defined.

Let σ denote the arc length of T_0 from O to Q, σ being reckoned positive or negative according as motion along T_0 from O to Q agrees or disagrees in sense with the motion of the electrons.

P lies in the normal plane Π of T_0 at Q. In Π we have the principal normal and the binormal of T_0 at Q. We assign positive directions on these normals, and on the tangent, in the usual way, so that the tangent, principal normal, and binormal form a right-handed system of axes. We employ the normals, thus sensed, as the axes of a rectangular coordinate system in Π, the principal normal being the ξ-axis, and the binormal being the η-axis. Let (ξ, η) be the coordinates of P in this system.

Then we take (σ, ξ, η) as the spacial coordinates of P. This coordinate system is adequate for the unambiguous representation of all points which are not too remote from T_0.

A simple and straightforward calculation shows that the differential distance ds between the points (σ, ξ, η) and $(\sigma + d\sigma, \xi + d\xi, \eta + d\eta)$ is given by the formula

$$ds^2 = [(1 - \xi\rho^{-1})^2 + (\xi^2 + \eta^2)\tau^{-2}]\,d\sigma^2 + d\xi^2 + d\eta^2 - 2\eta\tau^{-1}\,d\sigma\,d\xi + 2\xi\tau^{-1}\,d\sigma\,d\eta, \qquad (1)$$

where ρ and τ, respectively, denote the radius of curvature and the radius of torsion of T_0 at the point $(\sigma, 0, 0)$.

3. The system of differential equations defining the aggregate of trajectories in the neighborhood of T_0

The system of differential equations defining the totality of trajectories of electrons moving with energy E can be obtained from the principle of least action

[3] The latter condition is stated for the case in which we are employing classical dynamics. If we are using relativistic dynamics, the surface in question is represented by the equation $(E - V)^2 - m_0^2c^4 = 0$.

$$\delta \int_{P_1}^{P_2} \left[\Phi + A_1 \frac{d\sigma}{ds} + A_2 \frac{d\xi}{ds} + A_3 \frac{d\eta}{ds} \right] ds = 0,$$

where the A's are, except for a common constant factor, the components of the covariant vector potential of the electromagnetic field. Φ is a function of σ, ξ, η which has different forms depending upon whether we are employing classical dynamics or relativistic dynamics. In the first case

$$\Phi = (E - V)^{1/2};$$

in the second case

$$\Phi = [(E - V)^2 - m_0^2 c^4]^{1/2},$$

where m_0 is the rest-mass of an electron, and c is the speed of light.

We shall not carry out the actual computations here;[4] but it can be seen without difficulty that if we confine our attention to trajectories for which none of the variables ξ, η, $d\xi/d\sigma$, $d\eta/d\sigma$ becomes too large, the system of differential equations can be written in the form

$$\left.\begin{aligned} &\alpha_{11}(\sigma)\xi'' + \alpha_{12}(\sigma)\xi' + \alpha_{13}(\sigma)\xi + \alpha_{14}(\sigma)\eta'' + \alpha_{15}(\sigma)\eta' + \alpha_{16}(\sigma)\eta \\ &\qquad + (\text{terms of higher degree in } \xi, \eta, \xi', \eta') = 0, \\ &\alpha_{21}(\sigma)\xi'' + \alpha_{22}(\sigma)\xi' + \alpha_{23}(\sigma)\xi + \alpha_{24}(\sigma)\eta'' + \alpha_{25}(\sigma)\eta' + \alpha_{26}(\sigma)\eta \\ &\qquad + (\text{terms of higher degree in } \xi, \eta, \xi', \eta') = 0, \end{aligned}\right\} \tag{2}$$

where the primes denote differentiation with respect to σ. Since T_0 is itself a trajectory, the equations contain no terms independent of ξ, η, and their derivatives.

Because the arc of T_0 under consideration contains no point of the surface $E - V = 0$,[3] ∞^2 trajectories pass through each point on T_0, or sufficiently near to T_0. In particular, $\alpha_{11}(\sigma)\alpha_{24}(\sigma) - \alpha_{14}(\sigma)\alpha_{21}(\sigma)$ does not vanish for any of the values of σ under consideration.

We say that a trajectory (more strictly, an arc of a trajectory) lies in the neighborhood of T_0 if, and only if, for it the variables ξ, η, ξ', η' are all so small that terms of the second degree in these variable are negligible in comparison with terms of the first degree. We shall confine our attention to trajectories lying in the neighborhood of T_0, and hence we shall be concerned with the system of differential equations

$$\left.\begin{aligned} &\alpha_{11}(\sigma)\xi'' + \alpha_{12}(\sigma)\xi' + \alpha_{13}(\sigma)\xi + \alpha_{14}(\sigma)\eta'' + \alpha_{15}(\sigma)\eta' + \alpha_{16}(\sigma)\eta = 0, \\ &\alpha_{21}(\sigma)\xi'' + \alpha_{22}(\sigma)\xi' + \alpha_{23}(\sigma)\xi + \alpha_{24}(\sigma)\eta'' + \alpha_{25}(\sigma)\eta' + \alpha_{26}(\sigma)\eta = 0. \end{aligned}\right\} \tag{3}$$

[4] The calculations are given in considerable detail, in terms of somewhat different coordinates, by Cotte in the paper cited.

Any trajectory in the neighborhood of T_0 is represented approximately by a solution of equations (3); and, conversely, any solution of equations (3), for which ξ, η, ξ', η' are all small, represents approximately a trajectory in the neighborhood of T_0.

In the following, to avoid awkward and tedious circumlocutions, we shall carry on the discussion as though the system of equations (3) were strictly correct. It must be remembered, however, that the equations are only approximations to the correct equations (2), and that the conclusions arrived at may be only approximately true as regards the actual trajectories. The determination of the nature of the errors caused by the substitution of equations (3) for equations (2) is complicated and difficult, and will not be considered here.[5] Briefly, the errors are of the kind which arise in ordinary optical work when we neglect such quantities as squares and products of the angles between the axis of the instrument and the rays of light.

The system of equations (3) is a system of homogeneous linear equations of the fourth order. Consequently, if

$$\left.\begin{aligned}\xi &= \varphi_1(\sigma)\\ \eta &= \psi_1(\sigma)\end{aligned}\right\},\quad \left.\begin{aligned}\xi &= \varphi_2(\sigma)\\ \eta &= \psi_2(\sigma)\end{aligned}\right\},\quad \left.\begin{aligned}\xi &= \varphi_3(\sigma)\\ \eta &= \psi_3(\sigma)\end{aligned}\right\},\quad \left.\begin{aligned}\xi &= \varphi_4(\sigma)\\ \eta &= \psi_4(\sigma)\end{aligned}\right\},$$

are four independent particular solutions of the equations, the general solution is

$$\left.\begin{aligned}\xi &= C_1\varphi_1(\sigma) + C_2\varphi_2(\sigma) + C_3\varphi_3(\sigma) + C_4\varphi_4(\sigma),\\ \eta &= C_1\psi_1(\sigma) + C_2\psi_2(\sigma) + C_3\psi_3(\sigma) + C_4\psi_4(\sigma),\end{aligned}\right\} \tag{4}$$

where the C's are arbitrary constants. In accordance with what has been said above, we are really interested only in solutions for which all of the C's are small.

4. Trajectories joining two given points

Consider a point $P_1{:}(\sigma_1, \xi_1, \eta_1)$ in the plane $\sigma = \sigma_1$, and a point $P_2{:}(\sigma_2, \xi_2, \eta_2)$ in the plane $\sigma = \sigma_2 > \sigma_1$.[6] We ask if there are any trajectories joining these two points, and if so, how many.

A trajectory (4) joins the two points if, and only if, the constants C_n satisfy the equations

[5] Cotte gives some discussion of these errors in the paper cited.

[6] Of course, it is assumed that ξ_1, η_1, ξ_2, η_2 are small.

$$\left.\begin{aligned} C_1\varphi_{11} + C_2\varphi_{21} + C_3\varphi_{31} + C_4\varphi_{41} &= \xi_1\,, \\ C_1\psi_{11} + C_2\psi_{21} + C_3\psi_{31} + C_4\psi_{41} &= \eta_1\,, \\ C_1\varphi_{12} + C_2\varphi_{22} + C_3\varphi_{32} + C_4\varphi_{42} &= \xi_2\,, \\ C_1\psi_{12} + C_2\psi_{22} + C_3\psi_{32} + C_4\psi_{42} &= \eta_2\,, \end{aligned}\right\} \tag{5}$$

where $\varphi_{n1} = \varphi_n(\sigma_1)$, $\psi_{n1} = \psi_n(\sigma_1)$, $\varphi_{n2} = \varphi_n(\sigma_2)$, $\psi_{n2} = \psi_n(\sigma_2)$. Hence the answers to the above questions depend upon the properties of the system of equations (5), regarded as a system of algebraic equations in the unknowns C_n.

The properties of the system of equations (5) depend essentially on the rank of the matrix

$$M = \begin{Vmatrix} \varphi_{11} & \varphi_{21} & \varphi_{31} & \varphi_{41} \\ \psi_{11} & \psi_{21} & \psi_{31} & \psi_{41} \\ \varphi_{12} & \varphi_{22} & \varphi_{32} & \varphi_{42} \\ \psi_{12} & \psi_{22} & \psi_{32} & \psi_{42} \end{Vmatrix}.$$

This rank is independent of the choice of the set of fundamental solutions of the differential equations, and can only be 4, 3, or 2.[7] We shall discuss these cases in turn.

5. Case 1: rank of $M = 4$

In the case in which the rank of M is four, no matter what the values of ξ_1, η_1, ξ_2, η_2 may be, there is one and only one set of C's satisfying equations (5). Thus, given any point P_1 near T_0 in the plane $\sigma = \sigma_1$, and any point P_2 near T_0 in the plane $\sigma = \sigma_2$, there is just one trajectory, in the neighborhood of T_0, joining the two points. Through the point P_1 (or P_2) there pass in all ∞^2 trajectories in the neighborhood T_0. These trajectories intersect the plane $\sigma = \sigma_2$ (or $\sigma = \sigma_1$) in points which, in their totality, form a two-dimensional region.

6. Case 2: rank of $M = 3$

Now consider the case in which the rank of M is three.

We denote the left-hand members of equations (5), regarded as func-

[7] The statements are justified briefly by the fact (brought out gradually in the course of the subsequent discussion) that each of the possible ranks of M has characteristic geometrical consequences which are obviously independent of the choice of the fundamental solutions.

tions of the C's, by the symbols L_1, L_2, L_3, L_4, respectively. The rank of the matrix being three, we know that there exist multipliers μ_1, μ_2, μ_3, μ_4, which are not all zero, and which are functions of σ_1 and σ_2, but not of the C's, such that we have identically

$$\mu_1 L_1 + \mu_2 L_2 + \mu_3 L_3 + \mu_4 L_4 = 0.$$

Moreover, there is no other such linear relation between the L's which is independent of this.

This being so, a necessary and sufficient condition for the existence of a set of C's satisfying equations (5) is that ξ_1, η_1, ξ_2, η_2 shall satisfy the relation

$$\mu_1\xi_1 + \mu_2\eta_1 + \mu_3\xi_2 + \mu_4\eta_2 = 0. \tag{6}$$

If this condition is satisfied, the system of equations (5) has an infinity of particular solutions, and the general solution of the system contains one arbitrary constant.

If ξ_1 and η_1 are assigned fixed values, equation (6) represents a straight line in the plane $\sigma = \sigma_2$.[8] Thus, to the point P_1 in the plane $\sigma = \sigma_1$ there corresponds a straight line l_2 in the plane $\sigma = \sigma_2$. If P_2 is on l_2, there are ∞^1 trajectories passing through P_1 and P_2, and lying in the neighborhood of T_0. If P_2 is not on l_2, there is no trajectory, in the neighborhood of T_0, joining P_1 and P_2.

Again, if ξ_2 and η_2 are assigned fixed values, equation (6) represents a straight line in the plane $\sigma = \sigma_1$. Thus, to the point P_2 in the plane $\sigma = \sigma_2$ there corresponds a straight line l_1 in the plane $\sigma = \sigma_1$. If P_1 is on l_1, there are ∞^1 trajectories passing through P_1 and P_2, and lying in the neighborhood of T_0. If P_1 is not on l_1, there is no trajectory, in the neighborhood of T_0, joining P_1 and P_2.

To each point P_1 in the plane $\sigma = \sigma_1$ there corresponds a line l_2 in the plane $\sigma = \sigma_2$, and in general to different points there correspond different lines. However, we see that all of these lines are parallel, the common direction of the lines being determined by the ratio of μ_3 and μ_4.

Now let a be an arbitrary small constant. To any point P_1 of the line

$$l_1 : \mu_1\xi_1 + \mu_2\eta_1 = a \tag{7}$$

there corresponds the line

$$l_2 : \mu_3\xi_2 + \mu_4\eta_2 = -a. \tag{8}$$

[8] It is easily seen that equation (6) must contain effectively at least one of the variables of each of the sets (ξ_1, η_1), (ξ_2, η_2).

The line l_1 is intersected by ∞^3 trajectories in the neighborhood of T_0, ∞^2 of the trajectories passing through each point P_1 which is near T_0. The ∞^2 trajectories which pass through any particular point P_1 all intersect the line l_2, ∞^1 of the trajectories passing through each point P_2 (near T_0) of l_2. Thus, the ∞^3 trajectories, in the neighborhood of T_0, which intersect the line l_1 all intersect the line l_2, ∞^2 of the trajectories passing through each point P_2 near T_0. That is, to the line focus l_1 in the plane $\sigma = \sigma_1$ there corresponds the line focus l_2 in the plane $\sigma = \sigma_2$.

There are ∞^1 lines of the family (7), the different lines corresponding to different values of the parameter a. These lines are all parallel. To each of these lines there corresponds a different image in the plane $\sigma = \sigma_2$, the image being the corresponding line l_2 of the family (8). All of these image lines are parallel.

Of all the lines in the plane $\sigma = \sigma_1$ which pass near T_0, only those of the family (7) have images in the plane $\sigma = \sigma_2$. In fact, consider any such line l_1 in the plane $\sigma = \sigma_1$ which does not belong to the family (7). It is easily seen that the ∞^3 trajectories, in the neighborhood of T_0, which intersect l_1 intersect the plane $\sigma = \sigma_2$ in points which, in their totality, form a two-dimensional region.

Let a' be a small constant different from a. We consider the line (7) and its image (8), and also the line

$$l_1' : \mu_1\xi_1 + \mu_2\eta_1 = a'$$

and its image

$$l_2' : \mu_3\xi_2 + \mu_4\eta_2 = -a'.$$

The distance from the line l_1 to the line l_1' is $(a' - a)(\mu_1^2 + \mu_2^2)^{-1/2}$, and the distance from the line l_2 to the line l_2' is $-(a' - a)(\mu_3^2 + \mu_4^2)^{-1/2}$. Hence, it is appropriate to consider $-(\mu_1^2 + \mu_2^2)^{1/2}(\mu_3^2 + \mu_4^2)^{-1/2}$ as the magnification of the image with respect to the object.

Let us suppose, in order to make some parts of the preceding discussion more explicit, that the fundamental solutions are determined by the special initial conditions

$$\left.\begin{aligned}
&\varphi_1(\sigma_1) = 1, \quad \psi_1(\sigma_1) = \varphi_1'(\sigma_1) = \psi_1'(\sigma_1) = 0,\\
&\psi_2(\sigma_1) = 1, \quad \varphi_2(\sigma_1) = \varphi_2'(\sigma_1) = \psi_2'(\sigma_1) = 0,\\
&\varphi_3'(\sigma_1) = 1, \quad \varphi_3(\sigma_1) = \psi_3(\sigma_1) = \psi_3'(\sigma_1) = 0,\\
&\psi_4'(\sigma_1) = 1, \quad \varphi_4(\sigma_1) = \psi_4(\sigma_1) = \varphi_4'(\sigma_1) = 0.
\end{aligned}\right\} \tag{9}$$

Then the condition that M be of rank three is that we have the equation

$$\varphi_{32}\psi_{42} - \varphi_{42}\psi_{32} = 0, \tag{10}$$

and that φ_{32}, ψ_{32}, φ_{42}, ψ_{42} are not all zero.[9] We have the identities

$$\left.\begin{aligned} L_1 &\equiv C_1, \\ L_2 &\equiv C_2, \\ L_3 &\equiv \varphi_{12}C_1 + \varphi_{22}C_2 + \varphi_{32}C_3 + \varphi_{42}C_4, \\ L_4 &\equiv \psi_{12}C_1 + \psi_{22}C_2 + \psi_{32}C_3 + \psi_{42}C_4; \end{aligned}\right\} \tag{11}$$

and it readily follows from (10) and (11) that we can set

$$\mu_1 = \varphi_{12}\psi_{42} - \varphi_{42}\psi_{12}, \quad \mu_2 = \varphi_{22}\psi_{42} - \varphi_{42}\psi_{22}, \quad \mu_3 = -\psi_{42}, \quad \mu_4 = \varphi_{42},$$

or

$$\mu_1 = \varphi_{12}\psi_{32} - \varphi_{32}\psi_{12}, \quad \mu_2 = \varphi_{22}\psi_{32} - \varphi_{32}\psi_{22}, \quad \mu_3 = -\psi_{32}, \quad \mu_4 = \varphi_{32},$$

employing a set of μ's which are not all zero.

7. Case 3: rank of $M = 2$

Now consider the case in which the rank of M is two.

In this case there exist multipliers μ_{mn}, which are functions of σ_1 and σ_2, but not of the C's, such that we have the two independent identities

$$\mu_{11}L_1 + \mu_{12}L_2 + \mu_{13}L_3 + \mu_{14}L_4 = 0,$$
$$\mu_{21}L_1 + \mu_{22}L_2 + \mu_{23}L_3 + \mu_{24}L_4 = 0.$$

Moreover, there is no other linear homogeneous relation between the L's which is independent of the above.

A necessary and sufficient condition for the existence of a set of C's satisfying equations (5) is that ξ_1, η_1, ξ_2, and η_2 shall satisfy the two relations

$$\left.\begin{aligned} \mu_{11}\xi_1 + \mu_{12}\eta_1 + \mu_{13}\xi_2 + \mu_{14}\eta_2 &= 0, \\ \mu_{21}\xi_1 + \mu_{22}\eta_1 + \mu_{23}\xi_2 + \mu_{24}\eta_2 &= 0. \end{aligned}\right\} \tag{12}$$

If this condition is satisfied, the system of equations (5) has an infinity of particular solutions, and the general solution contains two arbitrary constants.

[9] We note that this condition has an obvious geometrical interpretation.

If ξ_1 and η_1 are assigned any fixed values, the equations (12) determine unique values of ξ_2 and η_2 ; and if any fixed values are assigned to ξ_2 and η_2 , the equations determine unique values of ξ_1 and η_1.[10] Thus, to each point P_1 in the plane $\sigma = \sigma_1$ there corresponds a unique point P_2 in the plane $\sigma = \sigma_2$, and conversely. If P_1 and P_2 are a pair of corresponding points near T_0 , all of the trajectories in the neighborhood of T_0 which pass through one of the points pass also through the other. We have a point-to-point image of a two-dimensional region of the one plane upon a two-dimensional region of the other.

Since the equations (12) are linear and homogeneous in the ξ's and the η's, the correspondence between the planes $\sigma = \sigma_1$ and $\sigma = \sigma_2$ is a centered affine point transformation. Hence, in order to determine the geometrical properties of the transformation, it is only necessary to determine the transforms of the points $(\sigma_1 , \epsilon, 0)$ and $(\sigma_1 , 0, \epsilon)$, where ϵ is a small positive constant.

If we write

$$\mu_{11}\mu_{24} - \mu_{14}\mu_{21} = D_1 , \qquad \mu_{13}\mu_{21} - \mu_{11}\mu_{23} = D_2 ,$$
$$\mu_{12}\mu_{24} - \mu_{14}\mu_{22} = D_3 , \qquad \mu_{13}\mu_{22} - \mu_{12}\mu_{23} = D_4 ,$$
$$\mu_{13}\mu_{24} - \mu_{14}\mu_{23} = D_0 ,$$

the transform of the point $(\sigma_1 , \epsilon, 0)$ has the coordinates

$$\xi_2 = -\epsilon D_1/D_0 , \qquad \eta_2 = -\epsilon D_2/D_0 , \tag{13}$$

and the transform of the point $(\sigma_1 , 0, \epsilon)$ has the coordinates

$$\xi_2 = -\epsilon D_3/D_0 , \qquad \eta_2 = -\epsilon D_4/D_0 . \tag{14}$$

The length of the vector (in the plane $\sigma = \sigma_2$) whose components are given by equations (13) is

$$\Lambda_1 = \epsilon \, | \, D_0 \, |^{-1}(D_1^2 + D_2^2)^{1/2},$$

the length of the vector whose components are given by equations (14) is

$$\Lambda_2 = \epsilon \, | \, D_0 \, |^{-1}(D_3^2 + D_4^2)^{1/2},$$

and the angle θ between the vectors is given by the formula

$$\cos\theta = \frac{D_1D_3 + D_2D_4}{[(D_1^2 + D_2^2)(D_3^2 + D_4^2)]^{1/2}} .$$

[10] If the determinant $\mu_{11}\mu_{22} - \mu_{12}\mu_{21}$ were zero, the only trajectories in the neighborhood of T_0 intersecting the plane $\sigma = \sigma_2$ would intersect that plane in a certain straight line, which is not the case. The assumption $\mu_{12}\mu_{24} - \mu_{14}\mu_{22} = 0$ leads to a similar contradiction.

In general, the image is rotated, magnified, and distorted, with respect to the object. The distortion is obviously characterized by the angle θ and by the ratio Λ_1/Λ_2. If, and only if, $\cos\theta$ is zero and Λ_1/Λ_2 is unity, there is no distortion.

The (areal) magnification is represented by ϵ^{-2} times the vector product of the vectors (13) and (14), in that order. Thus the magnification is

$$D_0^{-2}(D_1D_4 - D_2D_3).$$

(Since we are concerned only with vectors lying in a particular plane, the vector product is treated here as a mere scalar.)

If we employ the fundamental solutions defined by the conditions (9), the condition that M be of rank two readily reduces to the following form

$$\varphi_{32} = \psi_{32} = \varphi_{42} = \psi_{42} = 0.$$

The geometrical meaning of this is obvious. Also, when we employ these fundamental solutions we have the identities

$$L_1 \equiv C_1, \quad L_2 \equiv C_2, \quad L_3 \equiv \varphi_{12}C_1 + \varphi_{22}C_2, \quad L_4 \equiv \psi_{12}C_1 + \psi_{22}C_2,$$

so that we can set

$$\mu_{11} = \varphi_{12}, \quad \mu_{12} = \varphi_{22}, \quad \mu_{13} = -1, \quad \mu_{14} = 0,$$

$$\mu_{21} = \psi_{12}, \quad \mu_{22} = \psi_{22}, \quad \mu_{23} = 0, \quad \mu_{24} = -1.$$

8. Objects and images in three dimensions

So far we have been considering objects lying in a particular object plane $\sigma = \text{const}$. Now we are going to consider objects which do not satisfy this condition. It will be convenient to use σ_1 and σ_2 as variables, and to denote the particular values of these variables which we have been considering by the symbols σ_{10} and σ_{20}, respectively.

Let us denote the determinant of the matrix M by the symbol $\Delta(\sigma_1, \sigma_2)$.

In Case 1, $\Delta(\sigma_{10}, \sigma_{20})$ is not zero; and hence we see, by considerations of continuity, that M is of rank four for all pairs of values of σ_1 and σ_2 such that the value of σ_1 is sufficiently near σ_{10} and the value of σ_2 is sufficiently near σ_{20}. Therefore, we have the phenomena described in §5 for any such pair of values.

In Case 2, σ_{20} is a root of the equation

$$\Delta(\sigma_{10}, \sigma_2) = 0, \tag{15}$$

and is not a root of at least one of the equations

$$\Delta_{mn}(\sigma_{10}, \sigma_2) = 0, \qquad (m, n = 1, 2, 3, 4),$$

where the symbols $\Delta_{mn}(\sigma_1, \sigma_2)$ denote the several third-order minors of the determinant $\Delta(\sigma_1, \sigma_2)$.

In exceptional cases σ_{20} may conceivably be a multiple root of equation (15), and then we have a rather complicated situation which we shall not investigate here. In general, however, σ_{20} is a simple root of (15); and in this case, if σ'_{10} is a real value of σ_1 sufficiently near σ_{10}, the equation

$$\Delta(\sigma'_{10}, \sigma_2) = 0$$

has a simple real root σ'_{20} near σ_{20}, and at least one of the expressions $\Delta_{mn}(\sigma'_{10}, \sigma'_{20})$ is not zero.[11] Hence, in this case, to each plane $\sigma = \sigma'_{10}$ in the neighborhood of the plane $\sigma = \sigma_{10}$ there corresponds a plane $\sigma = \sigma'_{20}$ in the neighborhood of the plane $\sigma = \sigma_{20}$, such that for the planes $\sigma = \sigma'_{10}$ and $\sigma = \sigma'_{20}$, we have the phenomena described in §6. Thus, there is a congruence of straight lines in the object space (the neighborhood of the point $(\sigma_{10}, 0, 0)$) which is imaged line-to-line on a congruence of straight lines in the image space (the neighborhood of the point $(\sigma_{20}, 0, 0)$). Also, it is readily seen that no line in the object space not belonging to the first congruence has a line image in the image space.

The mapping of the one congruence upon the other deserves to be studied in detail; but that study will not be undertaken here.

In dealing with the case in which M is of rank two for $\sigma_1 = \sigma_{10}$ and $\sigma_2 = \sigma_{20}$, we shall employ the solutions determined by the initial conditions (9), where, however, σ_1 is now a variable parameter instead of a constant. Since the functions $\varphi_1(\sigma), \psi_1(\sigma), \cdots, \varphi_4(\sigma), \psi_4(\sigma)$ depend upon the parameter σ_1, we shall denote them henceforth by the symbols $\varphi_1(\sigma_1, \sigma), \psi_1(\sigma_1, \sigma), \cdots, \varphi_4(\sigma_1, \sigma), \psi_4(\sigma_1, \sigma)$.

In this case σ_{20} is a root of each of the equations

$$\varphi_3(\sigma_1, \sigma_2) = 0, \quad \psi_3(\sigma_1, \sigma_2) = 0, \quad \varphi_4(\sigma_1, \sigma_2) = 0, \quad \psi_4(\sigma_1, \sigma_2) = 0, \quad (16)$$

for $\sigma_1 = \sigma_{10}$. There are several conceivable special cases to be considered. These can all be treated along the same lines, and we shall discuss only a few of them explicitly. In particular, we shall suppose

[11] It is easily seen that $\Delta(\sigma_1, \sigma_{20})$ cannot vanish for all values of σ_1 near σ_{10}. Briefly, if it did, the ∞^2 trajectories passing through a certain line in the plane $\sigma = \sigma_{10}$ would all lie on a certain surface in the neighborhood of the point $(\sigma_{10}, 0, 0)$, whereas no surface can contain more than ∞^2 trajectories.

for the time being that no one of the functions $\varphi_3(\sigma_1, \sigma_{20})$, $\psi_3(\sigma_1, \sigma_{20})$, $\varphi_4(\sigma_1, \sigma_{20})$, $\psi_4(\sigma_1, \sigma_{20})$ vanishes for all values of σ_1 near σ_{10}.

Conceivably, σ_{20} may be a multiple root of one or more of equations (16) for $\sigma = \sigma_{10}$, and if this is so, we have a complicated situation which we shall not examine. In general, however, σ_{20} is a simple root of each equation for $\sigma = \sigma_{10}$; and if σ'_{10} is any real value of σ_1 near σ_{10}, each of the equations

$$\varphi_3(\sigma'_{10}, \sigma_2) = 0, \quad \psi_3(\sigma'_{10}, \sigma_2) = 0, \quad \varphi_4(\sigma'_{10}, \sigma_2) = 0, \quad \psi_4(\sigma'_{10}, \sigma_2) = 0$$

has a simple real root near σ_{20}. Let $\sigma'_{20}, \sigma''_{20}, \sigma'''_{20}, \sigma^{iv}_{20}$ denote these roots of the respective equations. We shall discuss the two extreme cases, namely: (A) the most general case, in which the four roots are distinct; and (B) the most special case, in which the four roots coincide.

A. It is easily seen that if the four roots $\sigma'_{20}, \sigma''_{20}, \sigma'''_{20}, \sigma^{iv}_{20}$ are distinct, no point in the plane $\sigma = \sigma'_{10}$ and near T_0 has a point image near T_0 and near the plane $\sigma = \sigma_{20}$. However, since σ_{20} is a root, and in fact at least a double root of the equation

$$\varphi_3(\sigma_{10}, \sigma_2)\psi_4(\sigma_{10}, \sigma_2) - \varphi_4(\sigma_{10}, \sigma_2)\psi_3(\sigma_{10}, \sigma_2) = 0,$$

the equation

$$\varphi_3(\sigma'_{10}, \sigma_2)\psi_4(\sigma'_{10}, \sigma_2) - \varphi_4(\sigma'_{10}, \sigma_2)\psi_3(\sigma'_{10}, \sigma_2) = 0 \tag{17}$$

may have certain real solutions near σ_{20}. If σ^v_{20} is such a solution, we have, for the planes $\sigma = \sigma'_{10}$, and $\sigma = \sigma^v_{20}$, the phenomena described in §6. If equation (17) has no real solutions near σ_{20}, no line in the plane $\sigma = \sigma'_{10}$ has an image in the image space.

B. If the four roots coincide, we have, for the planes $\sigma = \sigma_{10}$ and $\sigma = \sigma'_{20}$, the phenomena described in §7. The most important case is that in which the four roots coincide for all values σ'_{10} near σ_{10}, for in this case an entire three-dimensional neighborhood of the point $(\sigma_{10}, 0, 0)$ is imaged point-to-point on a three-dimensional neighborhood of the point $(\sigma_{20}, 0, 0)$. The remainder of the paper will be devoted to some further consideration of this case.

Let $P_1:(\sigma_1, \xi_1, \eta_1)$ and $P'_1:(\sigma_1 + d\sigma_1, \xi_1 + d\xi_1, \eta_1 + d\eta_1)$ be any two neighboring points of the object space, and let $P_2:(\sigma_2, \xi_2, \eta_2)$ and $P'_2:(\sigma_2 + d\sigma_2, \xi_2 + d\xi_2, \eta_2 + d\eta_2)$ be the corresponding points of the image space. By (1), the distance ds_1 between P_1 and P'_1 is given by the formula

$$ds_1^2 = [(1 - \xi_1\rho_1^{-1})^2 + (\xi_1^2 + \eta_1^2)\tau_1^{-2}]\, d\sigma_1^2 + d\xi_1^2 + d\eta_1^2 - 2\eta_1\tau_1^{-1}\, d\sigma_1 d\xi_1 + 2\xi_1\tau_1^{-1}\, d\sigma_1\, d\eta_1,$$

where ρ_1 and τ_1, respectively, denote the radius of curvature and the radius of torsion of T_0 at the point $(\sigma_1, 0, 0)$. In accordance with our procedure in the preceding sections, we assume that ξ_1 and η_1 are so small that terms of the second degree in these coordinates are negligible in comparison with terms of the first degree. Hence, we have approximately:

$$ds_1^2 = (1 - 2\xi_1\rho_1^{-1})\,d\sigma_1^2 + d\xi_1^2 + d\eta_1^2 - 2\eta_1\tau_1^{-1}\,d\sigma_1\,d\xi_1 + 2\xi_1\tau_1^{-1}\,d\sigma_1\,d\eta_1 . \quad (18)$$

Similarly, to the same approximation, the distance ds_2 between P_2 and P_2' is given by the formula

$$ds_2^2 = (1 - 2\xi_2\rho_2^{-1})\,d\sigma_2^2 + d\xi_2^2 + d\eta_2^2 - 2\eta_2\tau_2^{-1}\,d\sigma_2\,d\xi_2 + 2\xi_2\tau_2^{-1}\,d\sigma_2\,d\eta_2 ,$$

where ρ_2 and τ_2 denote the radius of curvature and the radius of torsion of T_0 at $(\sigma_2, 0, 0)$.

Now we have the equations

$$\left.\begin{aligned} \sigma_2 &= f(\sigma_1), \\ \xi_2 &= \varphi_1[\sigma_1, f(\sigma_1)]\xi_1 + \varphi_2[\sigma_1, f(\sigma_1)]\eta_1 \equiv \Phi_1(\sigma_1)\xi_1 + \Phi_2(\sigma_1)\eta_1 , \\ \eta_2 &= \psi_1[\sigma_1, f(\sigma_1)]\xi_1 + \psi_2[\sigma_1, f(\sigma_1)]\eta_1 \equiv \Psi_1(\sigma_1)\xi_1 + \Psi_2(\sigma_1)\eta_1 , \end{aligned}\right\} \quad (19)$$

where the first equation represents the solution under consideration of the system (16). It follows that we have the relations

$$\left.\begin{aligned} d\sigma_2 &= f'(\sigma_1)\,d\sigma_1 , \\ d\xi_2 &= [\Phi_1'(\sigma_1)\xi_1 + \Phi_2'(\sigma_1)\eta_1]\,d\sigma_1 + \Phi_1(\sigma_1)\,d\xi_1 + \Phi_2(\sigma_1)\,d\eta_1 , \\ d\eta_2 &= [\Psi_1'(\sigma_1)\xi_1 + \Psi_2'(\sigma_1)\eta_1]\,d\sigma_1 + \Psi_1(\sigma_1)\,d\xi_1 + \Psi_2(\sigma_1)\,d\eta_1 . \end{aligned}\right\} \quad (20)$$

Using (19) and (20), we can at once reduce the above formula for ds_2 to the form

$$ds_2^2 = g_{11}\,d\sigma_1^2 + g_{22}\,d\xi_1^2 + g_{33}\,d\eta_1^2 + 2g_{12}\,d\sigma_1\,d\xi_1 + 2g_{13}\,d\sigma_1\,d\eta_1 + 2g_{23}\,d\xi_1\,d\eta_1 , \quad (21)$$

where the g's are certain easily calculated functions of σ_1, ξ_1, η_1, which we do not need to exhibit explicitly.[12]

The geometrical properties of the mapping of the object space on the

[12] Naturally, in the calculation of the g's we ignore terms of the second and higher degrees in ξ_1 and η_1.

image space are determined in a familiar manner by the pair of formulae (18) and (21). We shall not discuss this in detail, but instead shall merely state, without proof, a few of the more important facts which are not already covered in the preceding work.

Suppose that $d\xi_1 = d\eta_1 = 0$. Then we have, to within terms of the second degree in ξ_1 and η_1,

$$\frac{ds_2}{ds_1} = f'(\sigma_1)\,[1 + (\rho_1^{-1} - \rho_2^{-1}\Phi_1)\,\xi_1 - \rho_2^{-1}\Phi_2\eta_1].$$

Consider the four points

$$P_1\!:(\sigma_1\,,\ \xi_1\,,\ \eta_1), \qquad P_1'\!:(\sigma_1 + d\sigma_1\,,\ \xi_1\,,\ \eta_1),$$

$$P_1''\!:(\sigma_1\,,\ \xi_1 + d\xi_1\,,\ \eta_1), \qquad P_1'''\!:(\sigma_1\,,\ \xi_1\,,\ \eta_1 + d\eta_1),$$

in the object space. We find that, to within terms of the second degree in ξ_1 and η_1, the cosine of the angle $P_1'P_1P_1''$ is $-\eta_1\tau_1^{-1}$ and the cosine of the image of this angle is

$$|f'|^{-1}(\Phi_1^2 + \Psi_1^2)^{-1/2}[(\Phi_1\Phi_1' + \Psi_1\Psi_1')\xi_1 + (\Phi_1\Phi_2' + \Psi_1\Psi_2' + \tau_2^{-1}f'\Phi_2\Psi_1 - \tau_2^{-1}f'\Phi_1\Psi_2)\eta_1].$$

Similarly, to the same approximation, the cosine of the angle $P_1'P_1P_1'''$ is $\xi_1\tau_1^{-1}$ and the cosine of the image of this angle is

$$|f'|^{-1}(\Phi_2^2 + \Psi_2^2)^{-1/2}[(\Phi_2\Phi_1' + \Psi_2\Psi_1' + \tau_2^{-1}f'\Phi_1\Psi_2 - \tau_2^{-1}f'\Phi_2\Psi_1)\xi_1 + (\Phi_2\Phi_2' + \Psi_2\Psi_2')\eta_1].$$

In the common case in which T_0 is the axis of an axially symmetric electrostatic field, we have[13]

$$\varphi_2(\sigma_1\,,\ \sigma_2) \equiv \psi_4(\sigma_1\,,\ \sigma_2), \qquad \psi_2(\sigma_1\,,\ \sigma_2) \equiv 0, \qquad \varphi_4(\sigma_1\,,\ \sigma_2) \equiv 0,$$

so the system of equations (16) is effectively equivalent to the single equation $\varphi_2(\sigma_1\,,\ \sigma_2) = 0$. Hence, except perhaps in very exceptional cases, if such a system images a particular object plane point-to-point on an image plane, it images an entire three-dimensional object space point-to-point on a three-dimensional image space. Most of the electron-optical systems in common use possess the same property, but apparently no radically different systems having the property are known. On the other hand, in ordinary (photon-) optics we know a variety of

[13] Of course, in order to treat this case by means of the present theory, the definitions in §2 must be supplemented by certain obvious conventions.

systems possessing the property that *all* rays which pass through an arbitrary point P_1 pass also through a corresponding point P_2. These are the so-called absolute optical instruments.[14] The simplest of them (aside from combinations of plane mirrors) is Maxwell's "fish's eye," which consists of an isotropic medium with the index of refraction $n = (n_0a^2)/(a^2 + r^2)$, where n_0 and a are constants, and r is the distance from a fixed point.

BELL TELEPHONE LABORATORIES
NEW YORK, N. Y.

[14] A contribution to the theory of the absolute optical instrument, and references to the extensive literature, are given in a paper by J. L. Synge, Trans. Am. Math. Soc., 44 (1938), pp. 32–46.

ON A CERTAIN NON-SEPARATING GRAPH ON AN ORIENTABLE SURFACE*

By I. N. Kagno

Introduction

Let G be a connected linear graph, containing no free arcs or chains,[1] which is mapped on an orientable surface Σ_p of genus $p > 0$, in such a manner that $\Sigma_p - G$ is a single 2-cell. It is known that G has at least $2p$ arcs, for we can choose for G a canonical system of simple closed curves on Σ_p, all passing through the same point x. Such a graph has $2p$ arcs and satisfies the conditions on G. In this paper we shall find an upper bound for the number of arcs in G, and determine the structure of G, for all possible choices of G. We shall then have an upper bound for the number of sides[2] of $\Sigma_p - G$. (The special case $p = 1$ was treated by the author in a previous paper, where he proved that for $p = 1$, $\Sigma_p - G$ has either four or six sides.[3]) This in turn enables us to place an upper bound on the number of sides of the fundamental domain of the group of decktransformations on the universal covering surface of Σ_p. When $p > 1$, the latter is closely related to the theory of Automorphic Functions and Uniformization theory.

1. Preliminary Definitions and Lemmas

Let G_1 be a graph consisting of two vertices a, b and three arcs, each of which has a for one endpoint and b for its other endpoint. Let G_2 be a graph consisting of a single vertex a together with two arcs, each of which has both of its endpoints coinciding with a. G_1 and G_2 are fundamental in this paper.

Let γ be a simple closed curve[4] mapped on a plane Π. By the *interior*

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[1] By a free arc or chain of a graph we mean an arc or chain one of whose endpoints is a vertex of degree 1.

[2] Defined on p. 87 of the author's paper "Perfect Subdivision of Surfaces." Journal of Math. and Physics, Vol. XVII (1938), pp. 76–111. We shall refer to this paper as PSS.

[3] PSS—Lemma 6.

[4] We shall use the abbreviation "scc" for the words "simple closed curve" in the sequel.

I_γ of γ we shall mean that portion of $\Pi - \gamma$ which does not contain the point at infinity. By $\bar{I}_\gamma$ we shall mean the closure of I_γ.

We shall say two graphs *meet* if they have points in common.

Lemma 1: *Let* γ_1 *and* γ_2 *be two scc which meet more than once, (i.e.* $\gamma_1 \cap \gamma_2$ *is not connected). If* $\gamma_1 + \gamma_2$ *is mapped on a plane* Π *in such a manner that* $\gamma_i - (\gamma_i \cap \gamma_j)$ *is wholly without* γ_j, $(i = 1, 2; j = 2, 1)$, *then* $\Pi - (\bar{I}_{\gamma_1} + \bar{I}_{\gamma_2})$ *is not connected.*

Proof: It will suffice to prove the lemma for the case where γ_1 meets γ_2 twice. $\gamma_1 \cap \gamma_2$ consists of either (a) two disjunct arcs rs and tu, or (b) an arc rs and a vertex $t \notin rs$, or (c) two distinct vertices r and t.

Case (a). I_{γ_1} and I_{γ_2} are each 2-cells whose boundaries have in common the arc rs. $I_{\gamma_1} + I_{\gamma_2} + rs$ is thus a 2-cell whose boundary touches itself along tu. Hence $R = I_{\gamma_1} + I_{\gamma_2} + rs + tu$ is a ring-shaped area and $\Pi - R$ is not connected. But $R = \bar{I}_{\gamma_1} + \bar{I}_{\gamma_2}$.

Case (b). This case arises from case (a) if tu is allowed to shrink to a point t. Obviously such an operation will not effect $\Pi - R$.

Case (c). This case arises from case (a) if we first allow tu to shrink to a point t, and then rs to shrink to a point r. This procedure obviously leaves $\Pi - R$ unchanged.

Lemma 2: *Let* γ_1 *and* γ_2 *be two scc which meet more than once, (i.e.* $\gamma_1 \cap \gamma_2$ *is not connected). If* $\gamma_1 + \gamma_2$ *is mapped on a plane* Π *in such a manner that* $\gamma_2 - (\gamma_1 \cap \gamma_2)$ *lies wholly within* γ_1, *then* $I_{\gamma_1} - \bar{I}_{\gamma_2}$ *is not connected.*

Proof: It will suffice to prove the lemma for the case where γ_1 meets γ_2 twice. $\gamma_1 \cap \gamma_2$ consists of either (a) two disjunct arcs rs and tu, or (b) an arc rs and a vertex $t \notin rs$, or (c) two distinct vertices r and t.

Case (a). Suppose the order of the vertices on γ_1 is r, s, t, u. $\gamma_2 - (rs + tu)$ consists two arcs ru and st. Now ru separates I_{γ_1} into two 2-cells I' and I''. st, except for its endpoints, is interior to one of these, say I'. But then st separates I' into two 2-cells I''' and J, where J is bounded by $rstur$ that is by γ_2. Thus $I'' + I''' = I_{\gamma_1} - J$ is not connected. But $J = \bar{I}_{\gamma_2}$.

The proof for cases (b) and (c) is similar.

Lemma 3: *Let* H_1 *and* H_2 *be two connected graphs, having* α_{11} *and* α_{12} *arcs respectively, which meet in such a manner that* $H_1 \cap H_2$ *is connected and at most a chain of* $H = H_1 + H_2$. *Then* $\alpha_1 \leq \alpha_{11} + \alpha_{12} + 3$, *where* α_1 *is the number of arcs in* H.

Proof: Case I, $H_1 \cap H_2$ is a chain uv of H.

uv is a subgraph of H_1 consisting of a chain containing α_{11}^* arcs, which either are all arcs of H_1, or else one or both of the end-arcs of uv are

subarcs of arcs of H_1. Similarly let α_{12}^* be defined for uv as a subgraph of H_2. As a subgraph of H, uv contains α^* arcs of H. $\alpha^* < \alpha_{11}^* + \alpha_{12}^*$.[5]

There arise various possibilities;

(1) u is a vertex a of H_1, and a vertex c of H_2. v is a vertex b of H_1 and a vertex d of H_2.

Then $H = (H_1 - uv) + (H_2 - uv) + uv$, and

$$\alpha_1 = (\alpha_{11} - \alpha_{11}^*) + (\alpha_{12} - \alpha_{12}^*) + \alpha^* < \alpha_{11} + \alpha_{12}.$$

(2) u is a vertex a of H_1 and a vertex c of H_2. v is a vertex b of H_1 but an interior point of an arc ef of H_2.

Then $\widehat{ef}$ of H_2 is the chain evf of H, and

$$H = (H_1 - uv) + [H_2 - (uv + vf)] + (uv + vf).$$

$$\therefore \alpha_1 = (\alpha_{11} - \alpha_{11}^*) + (\alpha_{12} - \alpha_{12}^*) + (\alpha^* + 1) < \alpha_{11} + \alpha_{12} + 1.$$

(3) u is a vertex a of H_1 but an interior point of an arc cd of H_2. v is a vertex b of H_1 but an interior point of an arc ef of H_2.

Then $\widehat{cd}$ of H_2 is the chain cud of H, $\widehat{ef}$ of H_2 is the chain evf of H, and

$$H = (H_1 - uv) + [H_2 - (uv + cu + vf)] + (uv + cu + vf).$$

$$\therefore \alpha_1 = (\alpha_{11} - \alpha_{11}^*) + (\alpha_{12} - \alpha_{12}^*) + (\alpha^* + 2) < \alpha_{11} + \alpha_{12} + 2.$$

(4) u is an interior point of an arc ab of H_1 and an interior point of an arc cd of H_2. v is a vertex b of H_1 but an interior point of an arc ef of H_2.

Then $\widehat{ab}$ of H_1 is the chain aub of H, $\widehat{cd}$ of H_2 is the chain cud of H, $\widehat{ef}$ of H_2 is the chain evf of H, and

$$H = [H_1 - (uv + au)] + [H_2 - (uv + cu + vf)] + (uv + au + cu + vf).$$

$$\therefore \alpha_1 = (\alpha_{11} - \alpha_{11}^*) + (\alpha_{12} - \alpha_{12}^*) + (\alpha^* + 3) < \alpha_{11} + \alpha_{12} + 3.$$

(5) u is an interior point of an arc ab of H_1 and an interior point of an arc cd of H_2. v is an interior point of an arc gh of H_1 and an interior point of an arc ef of H_2.

[5] For if uv as a subgraph of H_1 has α_{11}^* arcs, $(\alpha_{11}^* - 1)$ vertices of H_1 lie on uv. Likewise $(\alpha_{12}^* - 1)$ vertices of H_2 lie on uv. If no vertices of H_2 coincide with vertices of H_1 then uv, excluding its endpoints, contains $(\alpha_{11}^* + \alpha_{12}^* - 2)$ vertices of H, which separate it into $(\alpha_{11}^* + \alpha_{12}^* - 1)$ arcs. If one or more vertices of H_2 coincide with vertices of H_1 then uv, excluding its endpoints, has fewer than $(\alpha_{11}^* + \alpha_{12}^* - 2)$ vertices of H, and thus fewer than $(\alpha_{11}^* + \alpha_{12}^* - 1)$ arcs of H.

Then $\widehat{ab}$ of H_1 is the chain aub of H, $\widehat{gh}$ of H_1 is the chain gvh of H, $\widehat{cd}$ of H_2 is the chain cud of H, $\widehat{ef}$ of H_2 is the chain evf of H, and

$$H = [H_1 - (uv + au + vh)] + [H_2 + (uv + cu + vf)] + (uv + au + vh + cu + vf).$$

$$\therefore \alpha_1 = (\alpha_{11} - \alpha_{11}^*) + (\alpha_{12} - \alpha_{12}^*) + (\alpha^* + 4) < \alpha_{11} + \alpha_{12} + 4.$$

(6) u is a vertex a of H_1 and a vertex c of H_2. v is an interior point of an arc gh of H_1 and an interior point of an arc ef of H_2.

Then $\widehat{gh}$ of H_1 is the chain gvh of H, $\widehat{ef}$ of H_2 is the chain evf of H, and

$$H = [H_1 - (uv + vh)] + [H_2 - (uv + vf)] + (uv + vh + vf).$$

$$\therefore \alpha_1 = (\alpha_{11} - \alpha_{11}^*) + (\alpha_{12} - \alpha_{12}^*) + (\alpha^* + 2) < \alpha_{11} + \alpha_{12} + 2.$$

If chain uv of H is merely an arc or subarc of H_1 then $\alpha_{11}^* = 1$ in the above considerations. Similarly if chain uv of H is merely an arc or subarc of H_2 then $\alpha_{12}^* = 1$.

Case II, $H_1 \cap H_2$ is a vertex v of H.

If v is a vertex of H_1 but an interior point of an arc ab of H_2, then the arc ab of H_2 is the chain avb of H, and $H = H_1 + (H_2 - ab) + (av + vb)$. Hence $\alpha_1 = \alpha_{11} + (\alpha_{12} - 1) + 2 = \alpha_{11} + \alpha_{12} + 1$. Similarly if v is an interior point of an arc cd of H_1 but a vertex of H_2. If v is an interior point of an arc ab of H_2, and an interior point of an arc cd of H_1, it can be shown that $\alpha_1 = \alpha_{11} + \alpha_{12} + 2$. If v is a vertex of H_1 and a vertex of H_2, then $\alpha_1 = \alpha_{11} + \alpha_{12}$.

This completes the proof.

By a 0-*star* σ of a vertex v of degree ≥ 3 of a graph we shall mean the subgraph consisting of v, and all arcs of the graph incident with v together with their endpoints. By a *sub* 0-*star* of v we shall mean the subgraph consisting of v, and at least three, but not all of the arcs incident with v, together with their endpoints. By a *root* of a tree we shall mean a vertex of the tree which is of degree 1.

Lemma 4: *Let H be a graph homeomorphic with G_1, and K a connected graph, having α_{1k} arcs, which meets H, such that $K \cap H$ is connected. Let α_1 be the number of arcs in $K + H$. Then $\alpha_1 \leq \alpha_{1k} + 6$.*

Proof: $K \cap H$ is either (I) a chain, arc or vertex of $K + H$; or (II) a 0-star or sub 0-star of a vertex of $K + H$; or (III) a tree other than that in (I) and (II); or (IV) a circuit of $K + H$; or (V) a circuit γ together with an arc lm, $l \in \gamma$, $m \notin \gamma$.

Case I. If $K \cap H$ is a vertex, arc, or chain of $K + H$, then by lemma 3, since H has three arcs, $\alpha_1 \leq \alpha_{1k} + 6$.

Case II. $K \cap H$ is a 0-star σ, or sub 0-star of a vertex v of $K + H$. It is clear that v is also a vertex of K, and of H. Now each vertex of H is of degree three. Hence σ consists of v and the three arcs vv_1, vv_2, vv_3, where v_i is a point of the closure of some arc r_is_i of K. $(i = 1, 2, 3.)$ If v_i is not r_i or s_i then the arc r_is_i of K is the chain $r_iv_is_i$ of $K + H$. That is, r_is_i contributes two arcs to the total in $K + H$. Thus K considered as a subgraph of $K + H$ has at most $\alpha_{1k} + 3$ arcs of $K + H$. Since $H - (K \cap H)$ consists of three arcs, $K + H$ has at most $\alpha_{1k} + 6$ arcs.

Case III. $K \cap H$ is a tree other than that in cases I and II. Denote the roots of this tree by $v_1, \cdots, v_j$. Since this tree is a subgraph of H, and since the largest tree contained in H has four roots, $j \leq 4$. If all the v_i coincide with vertices of K, then K considered as a subgraph of $K + H$ has α_{1k} arcs of $K + H$. If v_i does not coincide with a vertex of K, it is an interior point of some arc r_is_i of K. Then r_is_i of K is the chain $r_iv_is_i$ of $K + H$ and contributes two arcs to the total in $K + H$. Similarly for the other roots. Hence K as a subgraph of $K + H$ has at most $\alpha_{1k} + j$ arcs of $K + H$. Clearly $H - (K \cap H)$ has two arcs. Hence $K + H$ has at most $\alpha_{1k} + j + 2 \leq \alpha_{1k} + 6$ arcs.

Case IV. $K \cap H$ is a circuit. In this case each vertex of H is a point on the closure of some arc of K. If the vertex v of H coincides with a vertex of K it does not effect the number of arcs in $K + H$, but if v is an interior point of an arc rs of K, then $\widehat{rs}$ of K is the chain rvs of $K + H$, and hence, since there are two vertices in H, K considered as a subgraph of $K + H$ has at most $\alpha_{1k} + 2$ arcs of H. $H - (K \cap H)$ consists of a single arc. Therefore $K + H$ has at most $\alpha_{1k} + 3$ arcs.

Finally, case V. If $K \cap H$ is the circuit γ together with an arc lm $l \in \gamma$, $m \notin \gamma$, it is easily seen that $K + H$ has at most $(\alpha_{1k} + 2) + 1 = \alpha_{1k} + 3$ arcs.

Lemma 5: Let H be a graph homeomorphic with G_2, and K a connected graph, having α_{1k} arcs, which meets H such that $K \cap H$ is connected. Let α_1 be the number of arcs in $K + H$. Then $\alpha_1 \leq \alpha_{1k} + 6$.

The proof is similar to that of lemma 4.

2. The Number of Arcs in G, and The Structure of G

The most general orientable surface of genus p is a sphere Σ_p with p handles $\mathfrak{H}_1, \cdots, \mathfrak{H}_p$. Let λ be a scc, mapped on Σ_p, which does not separate Σ_p. λ shall be said to *pass under* a handle $\mathfrak{H}$ of Σ_p if $\mathfrak{H}$ can

be picked off[6] leaving λ intact, and λ separates the resulting surface. λ shall be said to *pass over* $\mathfrak{H}$, if $\mathfrak{H}$ cannot be picked off without deleting a portion of λ.

Theorem 1: *G contains as subgraphs p graphs H_i, $(i = 1, \cdots, p)$, one for each handle of Σ_p, where H_i is homeomorphic with G_1 or with G_2. These H_i are not necessarily disjunct.*

Proof: G contains at least one circuit λ_1, for if G were a tree, $\Sigma_p - G$ would not be a 2-cell. λ_1 must either pass under or over some handle, call it $\mathfrak{H}_1$, for otherwise λ_1 would separate Σ_p. Suppose λ_1 passes under $\mathfrak{H}_1$. Then G must contain as subgraph an arc a_1b_1, $a_1 \epsilon \lambda_1$, $b_1 \epsilon \lambda_1$, which passes over $\mathfrak{H}_1$. For suppose this were not the case, then if Σ_p were cut along G it would open out into a region with at least two boundaries, each of these two boundaries being isomorphic with λ_1. But then $\Sigma_p - G$ could not be a 2-cell. Similarly if λ_1 passes over $\mathfrak{H}_1$, G must contain as subgraph an arc a_1b_1, $a_1 \epsilon \lambda_1$, $b_1 \epsilon \lambda_1$, which passes under $\mathfrak{H}_1$.

Let $H_1 = \lambda_1 + a_1b_1$. If $a_1 \neq b_1$ then H_1 is homeomorphic with G_1. If $a_1 = b_1$ then H_1 is homeomorphic with G_2.

Cut Σ_p along H_1. There results a Σ_{p-1} with a hole. Now $G - H_1$ contains at least one circuit λ_2, for if $G - H_1$ were a tree, then if Σ_{p-1} were cut along $G - H_1$ the resulting surface would not be a 2-cell, and hence $\Sigma_p - G$ would not be a 2-cell. λ_2 passes over or under some handle of Σ_{p-1}, call it $\mathfrak{H}_2$, for otherwise λ_2 would separate Σ_{p-1}, and then G would separate Σ_p. By reasoning similar to the preceding paragraphs we see that G contains an arc a_2b_2, $a_2 \epsilon \lambda_2$, $b_2 \epsilon \lambda_2$ which passes over (or under) $\mathfrak{H}_2$ according as λ_2 passes under (or over) $\mathfrak{H}_2$. Let $H_2 = \lambda_2 + a_2b_2$.

Proceeding in this manner we can show that G contains as subgraphs p such graphs H_i, $(i = 1, \cdots, p)$, one for each handle of Σ_p.

Theorem 2: *If the H_i are all disjunct, then $G - \sum_{i=1}^{p} H_i$ is a tree or set of trees having roots on $\sum H_i$ and no other roots, such that if each H_i is shrunk to a point G becomes a connected tree.*

Proof: Cut Σ_p along λ_i. There results a Σ_{p-1} with two holes, the edges λ_i', λ_i'' of these holes each being isomorphic with λ_i. Then a_ib_i has one of its endpoints on λ_i' and the other on λ_i''. For if a_ib_i had both its endpoints on λ_i' say, a_ib_i together with an arc of λ_i' would form a circuit μ which separates Σ_{p-1}. But then a_ib_i together with an arc of λ_i would form a circuit μ which separates Σ_p, contrary to fact. Cut

[6] The process of picking off a handle is intuitively obvious. See PSS—p. 83 *f.* for a definition.

Σ_{p-1} along $a_i b_i$. There results a surface Σ'_{p-1} with a single hole. If we continue in this manner, cutting Σ_p along each of the H_i there will result a sphere S with p holes.

If on S we allow each hole together with its edge to shrink to a point, (this procedure will not alter $G - \sum H_i$), there will result a sphere S^* on which is mapped the graph $G - \sum H_i$ having among its vertices the vertices $v_1, \cdots, v_p$ to which the various holes have been shrunk, where $\overline{G - \sum H_i}$ is the graph $G - \sum H_i$ together with the vertices v_i. (That is, $\overline{G - \sum H_i}$ can be obtained from G by allowing each H_i to shrink to a point v_i.) Since G is connected, $\overline{G - \sum H_i}$ is connected. Since $\Sigma_p - G$ is by hypothesis a 2-cell, $S - (G - \sum H_i)$ is a 2-cell, and hence when S^* is cut along $\overline{G - \sum H_i}$ there must result a single 2-cell. Hence $\overline{G - \sum H_i}$ cannot contain any circuits, for such a circuit would separate $S^* - (G - \sum H_i)$. Hence $\overline{G - \sum H_i}$ is a tree and has all of its roots contained among the vertices $v_1, \cdots, v_p$. For suppose the tree had a root u not contained among the v_i. Let uw be the arc of the tree which has u for end. Then uw is a free arc of $G - \sum H_i$, and hence of G. But by hypothesis G does not contain any free arcs.

That is, $G - \sum H_i$ is a tree or set of trees having the properties described in the statement of the theorem.

If in G the H_i are not all disjunct we have the

Theorem 3: *If H_1 meets H_2 then $H_1 \cap H_2$ is connected.*

Proof: For simplicity suppose that the H_j, $(j = 3, \cdots, p)$, are disjunct. Cut Σ_p along each H_j. There results a Σ_2 with $p - 2$ holes. Span each of these holes with a 2-cell $\alpha_2^{(j)}$ and let Σ_2^* denote the resulting surface. Now cut Σ_2^* along H_2. There results a Σ_1 with a hole. Let γ_2 denote the edge of the hole. Span this hole by a 2-cell $\alpha_2^{(2)}$ and let Σ_1^* denote the resulting surface. Cut Σ_1^* along H_1. There results a sphere S with a hole. Let γ_1 denote the edge of this hole. Since H_2 meets H_1, γ_2 meets γ_1, and does so as many times as H_2 meets H_1. Now $S^* = S - \alpha_2^{(2)}$ is homeomorphic with a planar region R bounded by $\gamma_1 + \gamma_2 \pmod 2$. Let Π denote the plane of R. γ_1 separates Π. If $\gamma_2 - (\gamma_1 \cap \gamma_2)$ is in the interior I_{γ_1} of γ_1, then R consists of $I_{\gamma_1} - \overline{I}_{\gamma_2}$, (where $\overline{I}$ is the closure of I). If γ_1 meets γ_2 more than once, then by lemma 2, R cannot be connected, hence $S - \alpha_2^{(2)}$ is not connected. But then surely $S - \sum_{k=1}^{p} \alpha_2^{(k)}$, which is the surface which results from cutting Σ_p along each of the H_i, is not connected. In other words, $\Sigma_p - \sum H_i$,

and thus $\Sigma_p - G$, is not connected, contrary to hypothesis. Hence γ_1 meets γ_2 exactly once, that is, $H_1 \cap H_2$ is connected.

If $\gamma_2 - (\gamma_1 \cap \gamma_2)$ is in the exterior E_{γ_1} of γ_1, then R consists of $\Pi - (\bar{I}_{\gamma_1} + \bar{I}_{\gamma_2})$. If γ_1 meets γ_2 more than once then by lemma 1, R cannot be connected. But then as above, $\Sigma_p - G$ is not connected. Hence γ_1 meets γ_2 exactly once, that is, $H_1 \cap H_2$ is connected.

Similarly any pair of H_i meets at most once. Suppose H_1 meets H_2, and some H_j, $(j \neq 1, 2)$, say H_3, meets $H_1 + H_2$. Then by reasoning similar to that in theorem 3 we can show that $H_3 \cap (H_1 + H_2)$ is connected. Likewise we can show that

Corollary: *If H_r, H_s, H_t, $\cdots$, H_u, H_v is a sequence of the H_i such that H_t meets $(H_r + H_s)$, $\cdots$, H_u meets $(H_r + H_s + H_t + \cdots)$, H_v meets $(H_r + H_s + H_t + \cdots + H_u)$, then $H_r \cap H_s$ is connected, $H_t \cap (H_r + H_s)$ is connected, $\cdots$, $H_u \cap (H_r + H_s + H_t + \cdots)$ is connected, and $H_v \cap (H_r + H_s + H_t + \cdots + H_u)$ is connected.*

Lemma 6: *If the H_i fall into x disjunct sets, $x \geq 1$, such that all those in the same set meet under the conditions set forth in the corollary to theorem* 3, *then if Σ_p is cut along $\sum H_i$, there results a sphere with x holes.*

Proof: Let the H_i fall into the sets $(H_1, H_2, \cdots, H_j)$, $(H_{j+1}, \cdots, H_k)$, $\cdots$, $(H_{l+1}, \cdots, H_p)$. Cut Σ_p along H_1. There results a Σ_{p-1} with a hole. Let γ_1 denote the edge of this hole. Span the hole by a 2-cell α_2' and cut the resulting surface along H_2. There results a Σ_{p-2} with a hole. Let γ_2 denote the edge of this hole. As was seen in the proof of theorem 3, γ_2 meets γ_1 exactly once, since H_1 meets H_2 exactly once. Hence if we delete α_2' from Σ_{p-2} there results a Σ_{p-2}^* with a single hole with edge $\gamma_1 + \gamma_2$ (mod 2). But Σ_{p-2}^* is precisely the surface that results if we cut Σ_p along $H_1 + H_2$. Proceeding in this manner we see that if Σ_p is cut along $\sum_{i=1}^{j} H_i$ there results a Σ_{p-j} with a single hole. Let γ denote the edge of this hole. Span the hole by a 2-cell α_2 and let Σ_{p-j}^* denote the resulting surface.

As in the preceding paragraph we can show that if Σ_{p-j}^* is cut along $\sum_{m=j+1}^{k} H_m$ there results a Σ_{p-k} with a single hole, with edge γ'. Since $\sum_{i=1}^{j} H_i$ is disjunct from $\sum_{m=j+1}^{k} H_m$, γ' does not meet γ. Hence if α_2 is deleted from Σ_{p-k} there results a Σ_{p-k}^* with two holes. But Σ_{p-k}^* is precisely the surface that results if Σ_p is cut along $\sum_{i=1}^{k} H_i$.

In like manner, we can show that if Σ_p is cut along $\sum_{i=1}^{p} H_i$ there results a sphere S with x holes.

Theorem 4: *G has at most $6p - 3$ arcs.*

Proof: Let H_i, $i = 1, \cdots, p$, be the subgraphs of G defined in theorem 1.

Case I. The H_i are all disjunct. In the proof of theorem 2 it was shown that if the H_i are all disjunct and if Σ_p is cut along each of the H_i, there results a sphere S with p holes. The structure of $G - \sum H_i$ must be such that when S is cut along $G - \sum H_i$ there results a single 2-cell. That is, cutting along $G - \sum H_i$ is equivalent to making a series of cross cuts from the edge of one hole to the edge of another. The first such cross cut we make will result in a sphere S' with $p - 1$ holes; each successive cross cut reducing the number of holes by one, and finally after the $(p - 1)$st cross cut there will result a sphere $S^{(p-1)}$ with a single hole. Each such cross cut is made along a chain of $G - \sum H_i$.

Let us count the number of arcs in $\sum H_i + M$, where M is the sum of the above mentioned chains. Consider the first cross cut. It is made along a chain uv of $G - \sum H_i$, where u and v are points on the edges of two distinct holes (and hence are points of $\sum H_i$). If u is a vertex of $\sum H_i$ it does not effect the number of arcs in G, but if u is an interior point of some arc rs of $\sum H_i$ then $\widehat{rs}$ of $\sum H_i$ is the chain rus of G, and $\widehat{rs}$ contributes two arcs to the total in G. Similarly for v. Hence $\sum H_i + uv$ has at most three more arcs than $\sum H_i$. Since each H_i has at most three arcs, $\sum H_i + uv$ thus has at most $3p + 3$ arcs. Now consider the second cross cut on S. It is made along a chain wx of $G - (\sum H_i + uv)$. Then by similar reasoning, $(\sum H_i + uv) + wx$ has at most three more arcs than $\sum H_i + uv$. That is, $\sum H_i + uv + wx$ has at most $3p + 2 \cdot 3$ arcs. $\cdots$ Proceeding in this way we see that $\sum H_i + M$ has at most $3p + (p - 1)3 = 6p - 3$ arcs.

We now show that $G - \sum H_i$ consists of M and no other points. For suppose $G - (\sum H_i + M) \neq 0$. After making the $(p - 1)$st cross cut we shall have obtained the sphere $S^{(p-1)}$ with a single hole. $G - (\sum H_i + M)$ cannot contain a circuit, for such a circuit, being mapped on $S^{(p-1)}$, would separate it, but then $\Sigma_p - G$ would not be connected. $G - (\sum H_i + M)$ contains no free arcs or chains, for if it did, G would contain free arcs or chains, contrary to hypothesis. Hence $G - (\sum H_i + M)$ must contain a chain yz having both its ends on $(\sum H_i + M)$. But then yz would be mapped on $S^{(p-1)}$ and have both its ends on the boundary of $S^{(p-1)}$ and hence would separate

$S^{(p-1)}$. But if this were so $\Sigma_p - G$ would not be connected. Hence $\sum H_i + M$ is identical with G, and G has at most $6p - 3$ arcs.

Case II. The H_i are not all disjunct. Suppose H_1 meets H_2. By theorem 3, $H_1 \cap H_2$ is connected. By lemmas 4, 5, $H_1 + H_2$ has at most $3 + 6 = 9$ arcs. If no other H_i meets $H_1 + H_2$, let K_1 denote $H_1 + H_2$. If some other H_i, say H_3, meets $H_1 + H_2$, then by the corollary to theorem 3, $H_3 \cap (H_1 + H_2)$ is connected. By lemmas 4, 5, $H_1 + H_2 + H_3$ has at most $9 + 6 = 3 + 2(6)$ arcs. Suppose H_4 meets $H_1 + \cdots + H_3, \cdots, H_j$ meets $H_1 + \cdots + H_{j-1}$, $j \leq p$. Let $K_1 = \sum_{n=1}^{j} H_n$. Then by the corollary to theorem 3 and lemmas 4, 5, the number of arcs in K_1 is at most $3 + 6(j - 1)$.

If $j = p$, then $K_1 = \sum_{i=1}^{p} H_i$ and K_1 is identical with G. For if Σ_p is cut along K_1 there results a sphere S with a single hole (lemma 6, with $x = 1$), and by reasoning as in the last paragraph of case I, (with S replacing $S^{(p-1)}$ and K_1 replacing $\sum H_i + M$ in the discussion) it follows that $K_1 = G$.

Suppose $j < p$. Then the H_i fall into x disjunct sets $(H_1, H_2, \cdots, H_j)$, $(H_{j+1}, \cdots, H_k), \cdots, (H_{l+1}, \cdots, H_p)$ such that those in the same set meet in the manner described in theorem 3 and its corollary, to form the connected subgraph K_s of G, $(s = 1, 2, \cdots, x)$. Then

K_1 has at most $3 + 6(j - 1)$ arcs,

K_2 has at most $3 + 6(k - j - 1)$ arcs,

$\cdots$

K_x has at most $3 + 6(p - l - 1)$ arcs,

and therefore $\sum_{s=1}^{x} K_s$ has at most $3x + 6(p - x)$ arcs.

Now consider $G - \sum K_s$. We can apply to it the same analysis that was applied to $G - \sum H_i$ in theorem 2, and in case I of the present theorem, noting that the K_s are all disjunct, and that if Σ_p is cut along $\sum K_s$ there results a sphere S^* with x holes (lemma 6). Doing this we find that $G - \sum K_s$ is a tree or set of trees, having roots on $\sum K_s$ and no other roots, (such that if each K_s is shrunk to a point there results a connected tree), and that G has at most $[3x + 6(p - x)] + (x - 1)3 = 6p - 3$ arcs.

This completes the proof.

We thus have an upper bound for the number of arcs in G. If for G we take a canonical system of non-separating scc on Σ_p, (i.e. we take

each H_i homeomorphic with G_2, and let all the H_i meet in a single point which is the unique vertex of each of the H_i,) then G has $2p$ arcs. Hence $2p$ is a lower bound for the number of arcs in G.

Now $\Sigma_p - G$ touches itself along each arc of G. Hence $\Sigma_p - G$ has two sides for each arc in G. That is, if β_1 denotes the number of sides, $4p \leq \beta_1 \leq 12p - 6$.

We summarize the results of this section in the

Theorem 5: *Let G be a connected graph, having no free arcs or chains, mapped on an orientable surface Σ_p of genus p in such a way that $\Sigma_p - G$ is a 2-cell. Then if α_1 is the number of arcs in G, $2p \leq \alpha_1 \leq 6p - 3$, and $\Sigma_p - G$ has β_1 sides, where $4p \leq \beta_1 \leq 12p - 6$. The structure of G is such that either*

(a) *G consists of p disjunct graphs H_i, each homeomorphic with G_1 or with G_2, together with a tree or set of trees having roots on $\sum H_i$ and no other roots, such that if each H_i is shrunk to a point G becomes a connected tree;*

or (b) *G consists of x disjunct connected subgraphs K_s, $(s = 1, \cdots, x, x \geq 1)$, together with a tree or set of trees having roots on $\sum K_s$ and no other roots, such that if each K_s is shrunk to a point G becomes a connected tree. Each K_s consists of a set of the H_i which meet in the manner described in theorem* 3 *and its corollary.*

3. Application to Uniformization Theory

The result we have just obtained is of interest because of its relation to Uniformization theory and Automorphic Functions.

Let U_p be the universal covering surface of Σ_p. Then U_p has on it a network G^* which covers G, and which is an infinite graph containing no free arcs or chains, and $U_p - G^*$ is an infinite set of 2-cells δ_i, each of which covers $\Sigma_p - G$. The group $\{t\}$ of decktransformations of U_p transforms G^* into itself, and carries δ_i into δ_j $(i \neq j)$. Hence δ_i together with a suitably chosen subset of the subgraph $F(\delta_i)$ of G^* which bounds δ_i, is a proper fundamental domain[7] of $\{t\}$.[8] Hence we have the

Corollary: *If $\mathfrak{D}$ is a proper fundamental domain of the group of decktransformations of the universal covering surface of an orientable surface of genus $p > 0$, and if β_1 is the number of sides of $\mathfrak{D}$, then $4p \leq \beta_1 \leq 12p - 6$.*

[7] Defined in PSS as a fundamental domain which consists of a 2-cell bounded by a graph K, having no free arcs or chains, together with a suitably chosen subset of K.

[8] PSS, p. 109.

If $p > 1$, U_p can be topologically mapped in the interior I of a circle of unit radius.[9] Then each δ_i of $U_p - G^*$ will be mapped into a simply-connected region $\mathfrak{D}_i$ of I, bounded by a scc γ_i having β_1 arcs, that is $\mathfrak{D}_i$ will have β_1 sides, and G^* will be mapped into an infinite graph D consisting of all points of each of the γ_i. We shall say two points of I are *congruent* if they are the images of congruent points of U_p. Corresponding to the group $\{t\}$ of decktransformations of U_p there is an isomorphic group $\{T\}$ of topological transformations of I into itself, each of which, except the identical transformation, leaves no point fixed, and carries every point of I into a congruent point, (and carries D into itself). Each $\mathfrak{D}_i$, together with a suitably chosen subset of its boundary is a proper fundamental domain of $\{T\}$.

In particular, if Σ_p is an algebraic Riemann surface of genus $p > 1$, then I is a single valued conformal map of U_p and $\{T\}$ is a group of single-valued conformal (i.e. linear) transformations of I into itself.[10] Hence from the corollary it follows that

If the universal covering surface of an algebraic Riemann surface of genus $p > 1$ is mapped conformally in the interior I of a circle, then a fundamental domain of the group $\mathfrak{G}$ of single valued conformal transformations of I into itself has β_1 sides, where $4p \leq \beta_1 \leq 12p - 6$.

The algebraic functions belonging to this Riemann surface may be expressed as single valued automorphic functions which belong to the group $\mathfrak{G}$.[11] We can thus place an upper bound on the number of sides of a fundamental domain of the group of linear transformations to which an automorphic function belongs.

4. Examples

When $p = 1$, the theorem 5 reduces to lemma 6 of the author's paper PSS.

As we have seen, the theorem is of special interest when $p > 1$. Let us consider the case $p = 2$. Then G has α_1 arcs, $4 \leq \alpha_1 \leq 9$, and $\Sigma_2 - G$ has β_1 sides, $8 \leq \beta_1 \leq 18$. From the theorem, it also follows that G has two subgraphs H_1, H_2 each homeomorphic with G_1 or G_2. If H_1 does not meet H_2 then $G - (H_1 + H_2)$ is a tree having one root on H_1 and one root on H_2 and no other roots, in other words an arc.

[9] B. v. Kerekjarto: "Vorlesungen über Topologie." Berlin, 1923, p. 179. The discussion which follows is a generalization of the remarks of Kerekjarto, pp. 179–80.

[10] Kerekjarto: loc. cit., p. 180.

[11] Ibid.

In plate I we illustrate all possible graphs G with H_1 disjunct from H_2. In plate II we illustrate all possible graphs G with $H_1 \leftrightarrow G_1$,[12] $H_2 \leftrightarrow G_2$ and H_1 meeting H_2. In plate III we illustrate all possible graphs G with $H_1 \leftrightarrow G_1$, $H_2 \leftrightarrow G_1$ and H_1 meeting H_2. In plate IV we illustrate all possible graphs G with $H_1 \leftrightarrow G_2$, $H_2 \leftrightarrow G_2$ and H_1 meeting H_2.

Because of the large number of possible choices of G, space does not permit giving examples of $\Sigma_p - G$ for every possible G. In figures 1 to 6 we give examples of $\Sigma_2 - G$ for one graph each of 4, 5, $\cdots$, 9 arcs. These examples serve to show that when $p = 2$, there exist fundamental domains of the transformations discussed in the previous section having 8, 10, $\cdots$, 18 sides. (In figs. 1–6, the edges are identified as indicated.)

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[12] The symbol $\leftrightarrow$ is used as an abbreviation for "is homeomorphic with."

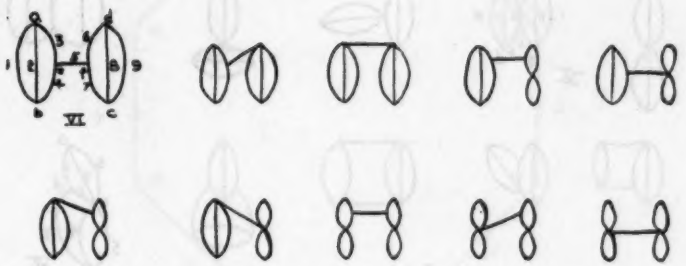

PLATE I. $H_1 \cap H_2 = 0$

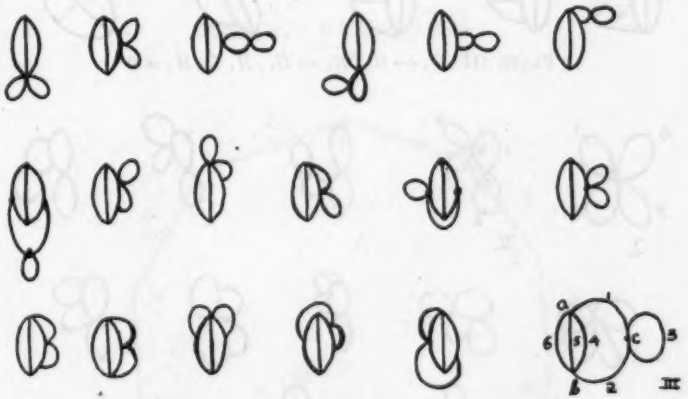

PLATE II. $H_1 \leftrightarrow G_1$, $H_2 \leftrightarrow G_2$, $H_1 \cap H_2 \neq 0$

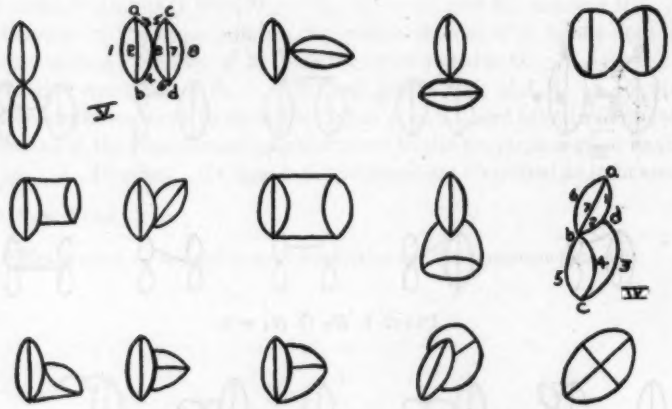

PLATE III. $H_1 \leftrightarrow G_1$, $H_2 \leftrightarrow G_1$, $H_1 \cap H_2 \neq 0$

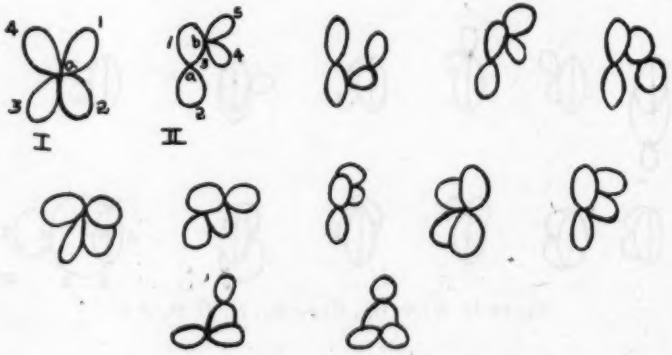

PLATE IV. $H_1 \leftrightarrow G_2$, $H_2 \leftrightarrow G_2$, $H_1 \cap H_2 \neq 0$

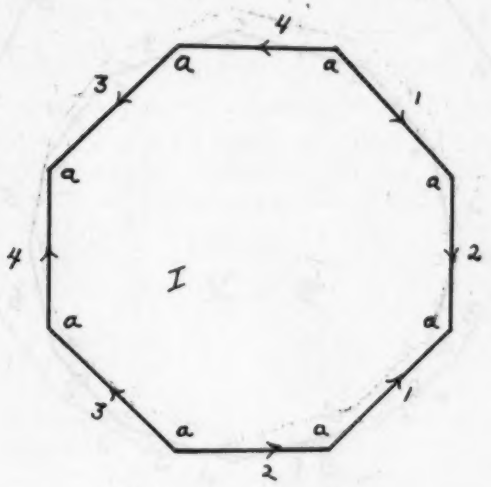

FIG. 1. $\Sigma_p - G$ has eight sides

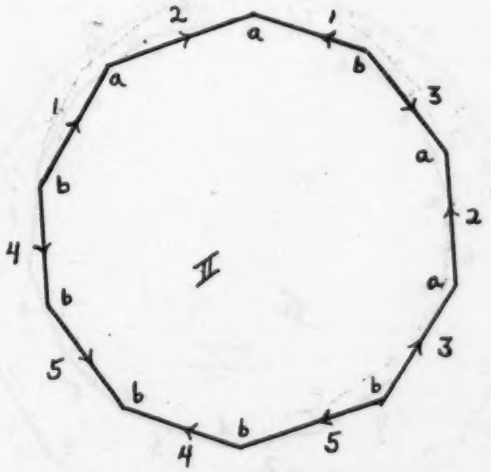

FIG. 2. $\Sigma_p - G$ has ten sides

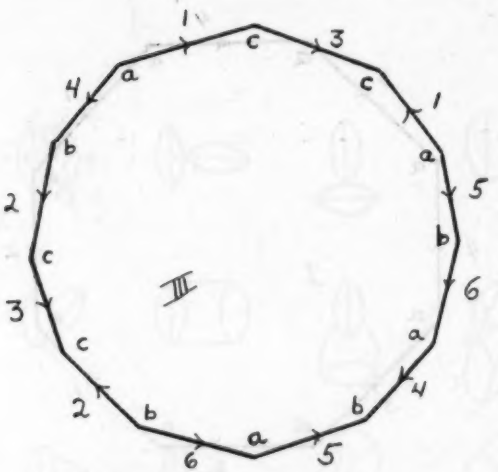

FIG. 3. $\Sigma_p - G$ has twelve sides

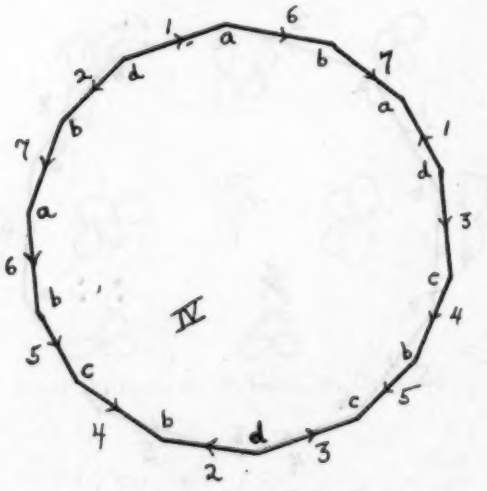

FIG. 4. $\Sigma_p - G$ has fourteen sides

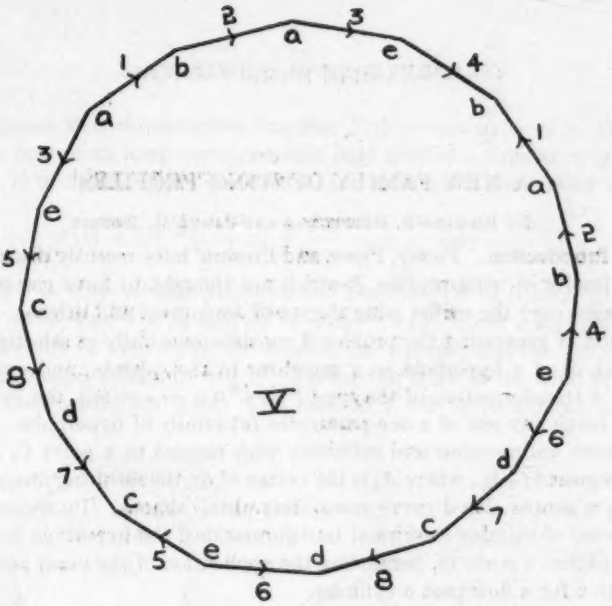

FIG. 5. $\Sigma_p - G$ has sixteen sides

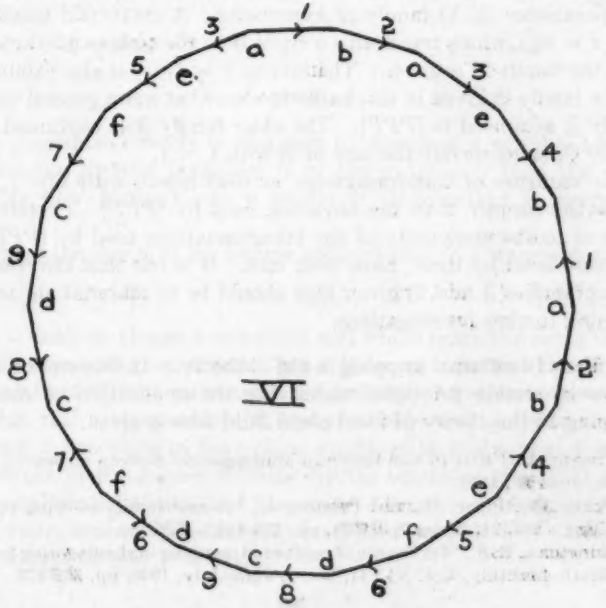

FIG. 6. $\Sigma_p - G$ has eighteen sides

A NEW FAMILY OF WING PROFILES[1]

By Richard S. Burington and James M. Dobbie

1. Introduction. Piercy, Piper, and Preston[2] have recently discussed a new family of wing profiles $\mathfrak{F}$ which are thought to have certain advantages over the earlier wing shapes of Joukowski and others. Their method of generating the profiles $\mathfrak{F}$ consists essentially in selecting one branch $\mathfrak{H}$ of a hyperbola as a generator in the z_δ-plane, and applying to $\mathfrak{H}$ a transformation of the type $\zeta = e^{i\beta}/(z_\delta + \epsilon + \delta i)$, the curve $\mathfrak{H}$ used being any one of a one-parameter (σ) family of hyperbolas. This produces an inversion and reflection with respect to a point C_δ in the line segment $\overline{C_\delta A_\delta}$, where A_δ is the vertex of $\mathfrak{H}$, the resulting image of $\mathfrak{H}$ being a simple closed curve resembling airfoil shapes. By means of a sequence of suitable conformal transformations, the hyperbola is transformed into a circle $\mathfrak{G}$, permitting the application of the usual potential function for a flow past a cylinder.

In the present paper a new family $\mathfrak{B}$ of curves resembling wing profiles is advanced in which the generating curve $\mathfrak{H}$ is any one of a general two-parameter (σ, λ) family of hyperbolas. A conformal transformation, $z = t(\zeta)$, which transforms a circle $\mathfrak{G}$ in the z-plane into any curve $\mathfrak{P}$ of the family $\mathfrak{B}$ is given. The inverse $\zeta = t^{-1}(z)$ is also exhibited.

The family $\mathfrak{B}$ given in this paper is somewhat more general than the family $\mathfrak{F}$ advanced in [*PPP*]. The older family $\mathfrak{F}$ is contained in the family $\mathfrak{B}$, being merely the case of $\mathfrak{B}$ with $\lambda = 1$.

The sequence of transformations, as used herein with $\lambda = 1$, seems somewhat simpler than the sequence used by [*PPP*]. Certain questions as to the analyticity of the transformations used by [*PPP*], but not considered by them, have been met. It is felt that the treatment of the families $\mathfrak{F}$ and $\mathfrak{B}$ given here should be of material aid to those pursuing further investigations.

2. Use of conformal mapping in airfoil theory. In this section a brief review of certain principles underlying the application of conformal mapping to the theory of ideal plane fluid flow is given.[3]

[1] Presented by title to the American Mathematical Society at Seattle, Washington, June 20, 1940.

[2] Piercy, N., Piper, R., and Preston, J., "A new family of wing profiles." Phil. Mag., Vol. 24, Series 7, (1937), pp. 425–444. [PPP].

[3] Burington, R. S., "On the use of conformal mapping in shaping wing profiles." Am. Math. Monthly, Vol. XLVII, No. 6, June–July, 1940, pp. 362–373.

Suppose that the potential function $F(z) = \phi(x, y) + i\psi(x, y)$ for a plane flow of an ideal incompressible fluid around a circular cylinder G (Fig. 1) of infinite span, center at z_0 and radius a in the z-plane is

$$F(z) = Ue^{-i\alpha}(z - z_0) + \frac{Ua^2e^{i\alpha}}{z - z_0} + \frac{i\Gamma}{2\pi}\log\frac{(z - z_0)}{a}, \tag{2.1}$$

where $z = x + iy$ and where on the boundary $\mathfrak{G}$ of G, $\psi(x, y) = 0$. Here $d\phi = u\,dx + v\,dy$ and $d\psi = u\,dy - v\,dx$, where u and v are the components of velocity of the flow in the z-plane; and the velocity of the air stream at infinity is $Ue^{i\alpha}$, α being the angle of the wind velocity with the positive real axis. The value of Γ will be considered later.

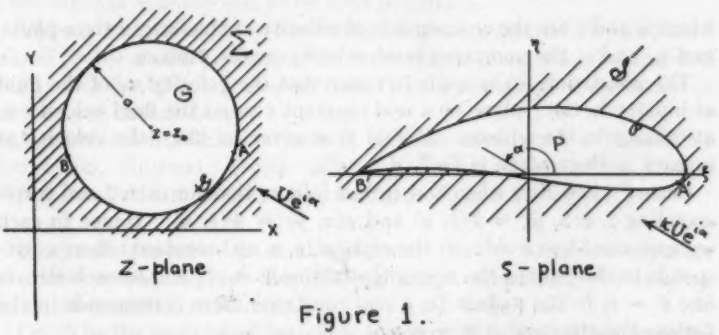

Figure 1.

The procedure consists in mapping by means of a suitable analytic function the circular boundary $\mathfrak{G}$ of the cylinder G (and the corresponding flow picture) into a profile P of boundary $\mathfrak{P}$ resembling airfoils.

Suppose the points in the z-plane are subjected to a transformation

$$z = t(\zeta), \qquad (\zeta = \xi + i\eta) \tag{2.2}$$

which is analytic almost everywhere and which maps the circle $\mathfrak{G}$ into a profile resembling an airfoil. Let $t(\zeta)$ be so restricted that: (a) to each point z of $\mathfrak{G}$ there corresponds a definite point ζ of the new profile $\mathfrak{P}$ and that this correspondence is biunique; (b) $dt/d\zeta$ is finite and not zero over Σ, the whole of the z-plane outside of $\mathfrak{G}$, and so that Σ will be represented by the domain $\mathfrak{S}$ of the ζ-plane outside of the airfoil $\mathfrak{P}$; (c) $t(\zeta)$ is analytic at infinity, and that as $\zeta \to \infty$, $z \to \infty$, and $dt/d\zeta \to \kappa$, a real, finite, non-zero constant; (d) everywhere outside of $\mathfrak{G}$, (2.2) has a single-valued inverse $\zeta = f(z)$.

The new potential function $\bar{F}(\zeta)$ for the flow about $\mathfrak{P}$ may be obtained from the potential $F(z)$ and transformation (2.2), and is given by

$$\bar{F}(\zeta) = F(t(\zeta)) = \Phi(\xi, \eta) + i\Psi(\xi, \eta). \tag{2.3}$$

Since for each z on $\mathfrak{S}$, $\psi(x, y) = 0$, and $F(z)$ is real, it follows from (2.3) and (a) that $\Psi(\xi, \eta) = 0$, for all points ζ on $\mathfrak{P}$, so that the flow across $\mathfrak{P}$ is zero.

The velocity function for the flow in the z-plane is $dF/dz = u - iv \equiv w_z$, and for the new flow in the ζ-plane is

$$\frac{d\bar{F}}{d\zeta} = \frac{dF}{dz}\frac{dz}{d\zeta} = w_z\frac{dt}{d\zeta} = u_\zeta - iv_\zeta \equiv w_\zeta, \tag{2.4}$$

where u and v are the components of velocity of the flow in the z-plane, and u_ζ and v_ζ the components of velocity in the ζ-plane.

The restriction (c) is made in order that the velocity w_ζ of the fluid at infinity in the ζ-plane be a real constant κ times the fluid velocity w_z at infinity in the z-plane. If $F(z)$ is as given in (2.1), the velocity at infinity in the z-plane is $Ue^{i\alpha}$.

Since $\bar{F}(\zeta) = F(z)$ when $z = t(\zeta)$, it follows that for each ζ and corresponding z, $\phi(x, y) = \Phi(\xi, \eta)$ and $\psi(x, y) = \Psi(\xi, \eta)$. Hence to each equipotential line $\phi = c_1$ in the z-plane (c_1 a real constant) there corresponds in the ζ-plane the equipotential line $\Phi = c_1$, and to each stream line $\psi = c_2$ in the z-plane (c_2 a real constant) there corresponds in the ζ-plane the stream line $\Psi = c_2$.

The restriction in (b) that $dt/d\zeta$ shall be analytic, finite, and non-zero over Σ is made in order that the flow pattern in the region Σ of the z-plane formed by the families $\phi = c_1$, $\psi = c_2$ shall be mapped conformally (with preservation of angles) into the flow pattern formed by the families $\Phi = c_1$ and $\Psi = c_2$ in the ζ-plane. Since $\phi = c_1$ and $\psi = c_2$ are orthogonal, so are $\Phi = c_1$ and $\Psi = c_2$.

The points in the ζ-plane where $d\bar{F}/d\zeta = 0$ are called the *critical points* of the ζ-plane; at such points the velocity w_ζ vanishes. From (2.4), it follows that if $dt/d\zeta$ is finite and non-zero at a critical point A' in the ζ-plane, then $dF/dz = 0$, and the point A in the z-plane corresponding to A' is a critical point of flow in the z-plane.

If, however, B is a critical point in the z-plane and at B $d\zeta/dz$ is zero, then the second member of (2.4) is indeterminate. Let B' be the point in the ζ-plane corresponding to B. The velocity of flow at B' is required to be finite and non-zero, which implies that $d\bar{F}/d\zeta$ is finite and non-zero at B'. (This condition is known as the Joukowski condition

for "smooth flow" at the trailing edge.) Evidently, B' is not a critical point in the ζ-plane.

Let $z_B = z_0 + ae^{-i\chi}$ be the point B on the cylinder $\mathfrak{G}$ corresponding to B', the trailing edge of the profile $\mathfrak{P}$. From the Joukowski condition, it follows that

$$\left.\frac{dF}{dz}\right]_B = Ue^{-i\alpha} - Ue^{i(\alpha+2\chi)} + \frac{i\Gamma}{2\pi a}e^{i\chi} = 0, \tag{2.5}$$

hence

$$\Gamma = 4\pi Ua \sin(\alpha + \chi). \tag{2.6}$$

This value of Γ is the one to be used in (2.1).

This shows how one may obtain from a given potential $F(z)$ and associated profile $\mathfrak{G}$, the potential $\bar{F}(\zeta)$ for the flow of an ideal fluid around any profile $\mathfrak{P}$ which is generated from $\mathfrak{G}$ by a conformal transformation of the type described.

A number of families of airfoil sections have been advanced by Joukowski, Karman, Treftz, Mises, Hohndorf, Theodorsen, Piercy, Piper, Preston, and others, using the methods just described.

3. **Airfoils generated from a hyperbola.** In this section a sequence of transformations which map a circle into a closed curve resembling an airfoil section is exhibited.

Let $\mathfrak{H}$ be the right-hand branch of the hyperbola (Fig. 3) whose equation in the z_5-plane is

$$\frac{x_5^2}{\lambda^2\cos^2\sigma\pi} - \frac{y_5^2}{\lambda^2\sin^2\sigma\pi} = 1, \tag{3.1}$$

where λ and σ are real and positive, and $0 < \sigma < \frac{1}{2}$. Let $z_5 = x_5 + iy_5$. Let $(-\epsilon - \delta i)$ be a point C_5 in the region to the left of $\mathfrak{H}$. The transformation

$$\zeta = \frac{e^{i\beta}}{z_5 + \epsilon + \delta i}, \tag{3.2}$$

where $\tan\beta = \delta/(\lambda\cos\sigma\pi + \epsilon)$ and $\zeta = \xi + i\eta$, produces an inversion of z_5 with respect to C_5 followed by a rotation of angle β. The image of $\mathfrak{H}$ in the ζ-plane is a closed curve $\mathfrak{P}$ resembling an airfoil section. (Fig. 2.) As a point P_5 on $\mathfrak{H}$ passes downward from ∞ to A_5 and on to ∞, the image P of P_5 in the ζ-plane moves from B counterclockwise

around the boundary of the profile shown in Fig. 2 to A and on around to B. The images of A_5, ∞, C_5 are A, B, ∞, respectively. The entire region to the left of $\mathfrak{H}$ maps into the entire region exterior to $\mathfrak{P}$.

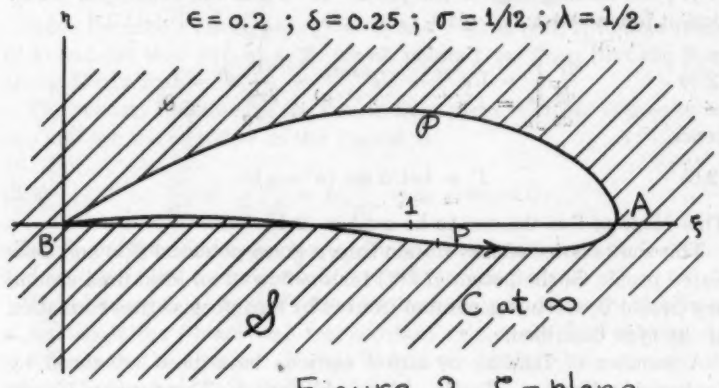

Figure 2. ζ-plane.

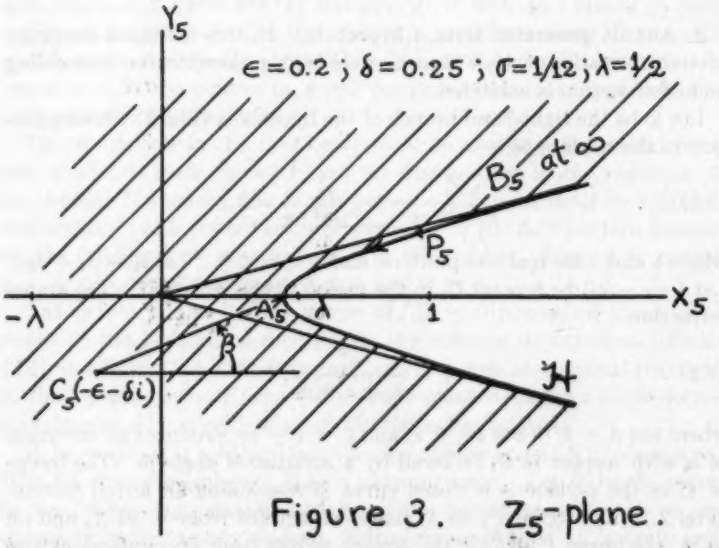

Figure 3. Z_5-plane.

In the following paragraphs a sequence of transformations which maps $\mathfrak{H}$ into a circle is exhibited.

The transformation

$$z_5 = \lambda \cosh z_4 , \tag{3.3}$$

where λ is real, non-zero and positive, and $z_4 = x_4 + iy_4$, maps the z_4-plane into the z_5-plane. $\mathfrak{H}$ is the image in the z_5-plane of the line $y_4 = \sigma\pi$ in the z_4-plane. As a point P_5 on $\mathfrak{H}$ passes downward from ∞ to A_5 and on to ∞, the image P_4 in the z_4-plane of P_5 moves from ∞ to the left along the line $y_4 = \sigma\pi$ to A_4 and on to the left to ∞. (Fig. 4.) The line $y_4 = \pi$ is the image of that part of the x_5-axis lying to the left of $z_5 = -\lambda$, inclusive. The region to the left of $\mathfrak{H}$ is mapped by (3.3) into the region defined by $\sigma\pi < y_4 \leqq \pi$. In order that (3.3)

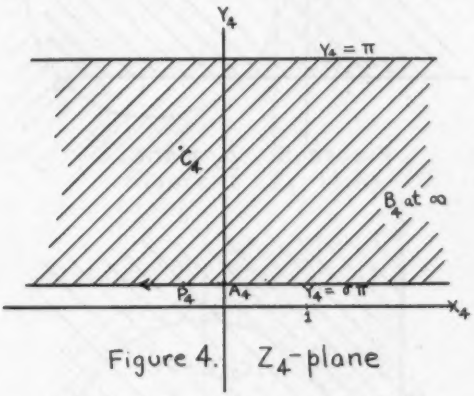

Figure 4. Z_4-plane

have a single-valued inverse, the strip for which $0 \leqq y_4 \leqq \pi$ is selected as a fundamental region in the z_4-plane.

The transformation

$$z_4 = \pi i - 2(1 - \sigma)z_3 \tag{3.4}$$

translates, reflects, and shrinks the cross-hatched region of the z_4-plane into the cross-hatched region shown in the z_3-plane. (Fig. 5.) As a point P_4 moves to the left along $y_4 = \sigma\pi$, the image P_3 moves to the right along $y_3 = \pi/2$.

The cross-hatched strip of the z_3-plane is mapped into the right half of the z_2-plane by the transformation

$$z_2 = \cosh z_3 . \tag{3.5}$$

The strip for which $0 \leqq y_3 \leqq \pi$ is selected as the fundamental region in the z_3-plane. As P_3 moves to the right along $y_3 = \pi/2$, the image P_2 in the z_2-plane moves upward along the y_2-axis. (Fig. 6.)

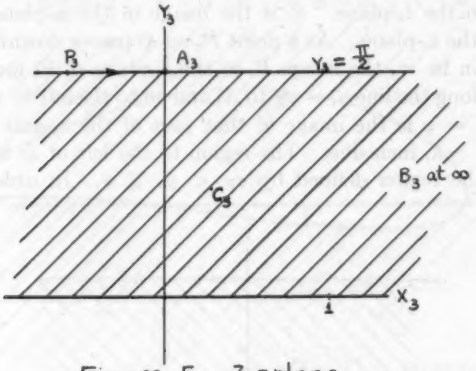

Figure 5. Z_3-plane.

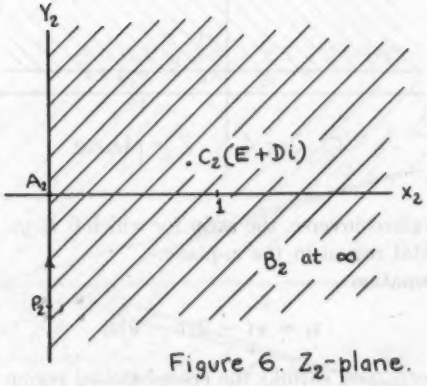

Figure 6. Z_2-plane.

The image C_2 of C_3 is given by

$$E + Di = \cosh\left[\frac{\cosh^{-1}(\epsilon + \delta i)/\lambda}{2(1-\sigma)}\right]. \tag{3.6}$$

Here $E > 0$ since C_2 is an interior point of the cross-hatched region of the z_2-plane.

Transformation

$$z_2 = E + Di + \frac{2E}{z_1} \tag{3.7}$$

maps the right half of the z_2-plane into the entire region outside the unit

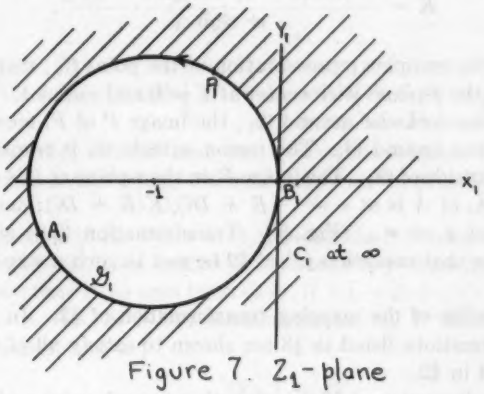

Figure 7. Z_1-plane

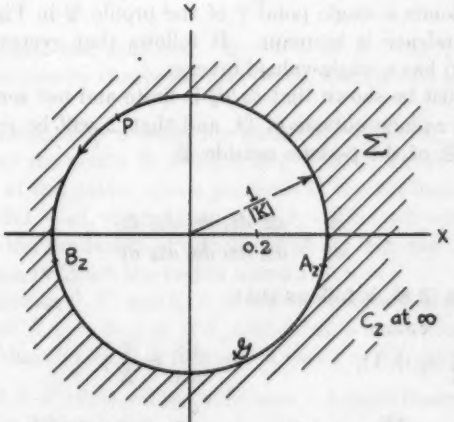

Figure 8. Z-plane

circle with center at $z_1 = -1$. (Fig. 7.) As a point P_2 moves upward along the y_2-axis, its image P_1 in the z_1-plane moves counterclockwise once around the circle $\mathfrak{G}_1$. The image C_1 of C_2 in the z_1-plane is at ∞.

The transformation

$$z_1 = Kz - 1, \tag{3.8}$$

where

$$K = \frac{-4E\lambda(1-\sigma)\sinh[2(1-\sigma)c_3]}{e^{i\beta}\sinh c_3}, \tag{3.9}$$

where c_3 is the complex representation of the point C_3, maps $\mathfrak{G}_1$ into a circle $\mathfrak{G}$ in the z-plane with center at $z = 0$ and radius $1/|K|$. As P_1 moves counterclockwise around $\mathfrak{G}_1$, the image P of P_1 moves counterclockwise once around $\mathfrak{G}$. The region outside $\mathfrak{G}_1$ is transformed into the region outside of $\mathfrak{G}$. The image B_z in the z-plane of B is at $z = 1/K$; the image A_z of A is at $z = (-E + Di)/K(E + Di)$; and the image C_z of C is at $z = \infty$. (Fig. 8.) Transformation (3.8) is introduced here in order that condition (d) of §2 be met in such a way that $\kappa = 1$.

4. **Application of the mapping transformation of §3.** In this section the transformations listed in §3 are shown to satisfy all of the requirements listed in §2.

From the discussion of §3 it is seen that to each point z of $\mathfrak{G}$ in Fig. 8 there corresponds a single point ζ of the profile $\mathfrak{P}$ in Fig. 2 and that this correspondence is biunique. It follows that everywhere outside of $\mathfrak{G}$, $\zeta = f(z)$ has a single-valued inverse.

Next, it must be shown that $dz/d\zeta$ is finite and not zero over Σ, the whole of the z-plane outside of $\mathfrak{G}$, and that Σ will be represented by the domain $\mathfrak{S}$ of the ζ-plane outside $\mathfrak{P}$.

Now

$$\frac{dz}{d\zeta} = \frac{dz}{dz_1}\frac{dz_1}{dz_2}\frac{dz_2}{dz_3}\frac{dz_3}{dz_4}\frac{dz_4}{d\zeta}. \tag{4.1}$$

From (3.2) to (3.8),,it follows that

$$z = \frac{1}{K}(z_1 + 1), \qquad \frac{dz}{dz_1} = \frac{1}{K}; \tag{4.2}$$

$$z_1 = \frac{2E}{z_2 - (E + Di)}, \qquad \frac{dz_1}{dz_2} = \frac{-2E}{[z_2 - (E + Di)]^2}; \tag{4.3}$$

$$z_2 = \cosh z_3, \qquad \frac{dz_2}{dz_3} = \sinh z_3; \tag{4.4}$$

$$(4.5)\quad z_3 = -\lambda \cosh[2(1-\sigma)z_3], \qquad \frac{dz}{dz_3} = \frac{-1}{2\lambda(1-\sigma)\sinh[2(1-\sigma)z_3]},$$

$$(4.6)\quad z_3 = \frac{e^{i\beta}}{\zeta} - \epsilon - \delta i, \qquad \frac{dz_3}{d\zeta} = \frac{-e^{i\beta}}{\zeta^2};$$

and

$$(4.7)\quad \zeta = \frac{e^{i\beta}}{\epsilon + \delta i - \lambda \cosh\left[2(1-\sigma)\cosh^{-1}\left(E + Di + \frac{2E}{Kz-1}\right)\right]}.$$

Evidently, $dz/d\zeta$ is finite and $\neq 0$ everywhere outside the circle $\mathfrak{G}$, except possibly at the points where the derivatives in (4.1) are zero or infinite.

An examination of the terms in (4.1) yields the following information:

The derivative dz/dz_1 is finite and not zero everywhere in Σ, provided $K \neq 0$, or ∞; that is, as seen from (3.9), if $-\epsilon - \delta i \neq -\lambda$. However, if $-\epsilon - \delta i = -\lambda$, K is indeterminate. As $-\epsilon - \delta i \to -\lambda$, the limit of K exists, and is

$$(4.8)\qquad K' = \frac{-8E\lambda(1-\sigma)^2}{e^{i\beta}},$$

where K' is finite and non-zero. If $-\epsilon - \delta i = -\lambda$, K is defined to be K'. It can now be concluded that dz/dz_1 is finite and nonzero everywhere in Σ.

Further examination reveals the possibility of $dz/d\zeta$ being zero or infinite only at the points B, B being at $\zeta = 0$; C, which is at ∞ in the ζ-plane; and at the points whose positions in the z_3-plane are $z_3 = \pi i j$, and $z_3 = \pi i j/2(1-\sigma)$, where j is an integer. (The only such values of z_3 which fall in the fundamental region of the z_3-plane and which are the image of points in $\mathfrak{S}$ are the points where $j = 0$.)

Next, the points B, C, and $z_3 = 0$ will be examined in some detail.

At the point B, $\zeta = 0$, $z = 1/K$, and $dz/d\zeta$ is not defined. A calculation shows that $\lim_{\zeta \to 0} dz/d\zeta$ is infinite, so that $z = t(\zeta)$ is not conformal at $\zeta = 0$. At B, $d\zeta/dz$ is defined to be zero. A more thorough examination of B will be made later.

At the point C, $z = \zeta = \infty$, and

$$(4.9)\qquad \lim_{\zeta \to \infty} \frac{dz}{d\zeta} = \frac{-4E\lambda(1-\sigma)}{Ke^{i\beta}} \frac{\sinh[2(1-\sigma)c_3]}{\sinh c_3},$$

where $c_3 = \cosh^{-1}(E + Di)$. This limit will be equal to 1 if K is selected to be the value stated in (3.9).

At $z_3 = 0$,

$$\lim_{z_3 \to 0} \frac{dz}{d\zeta} = \frac{-E(\lambda - \epsilon - \delta i)^2}{2e^{i\beta}(1-\sigma)^2 K\lambda(1 - E - Di)^2}. \tag{4.10}$$

This limit will be finite and non-zero if $E + Di \neq 1$. It can be shown that $\epsilon + \delta i \neq \lambda$, when $E + Di \neq 1$, and conversely. If $E + Di = 1$, $\lim_{z_3 \to 0} dz/d\zeta$ is defined to be the limit of the right hand member of (4.10) as $E + Di \to 1$. This limit is 1.

It has now been shown that all of the requirements (a), (b), (c), and (d) of §2 are met by the transformation (4.7). The inverse of (4.7) is

$$z = \frac{1}{K}\left\{1 + \frac{2E}{\cosh\left[\frac{1}{2(1-\sigma)}\cosh^{-1}\left\{\frac{1}{\lambda}\left(\epsilon + \delta i - \frac{e^{i\beta}}{\zeta}\right)\right\}\right] - (E + Di)}\right\}. \tag{4.11}$$

The point B, at $\zeta = 0$, is selected as the trailing edge, the tail point. The image of B in the z-plane is at $z = 1/K$. At B, $d\zeta/dz = 0$. The Joukowski condition requires that $d\bar{F}/d\zeta$ be finite and non-zero at B. From (2.4), it follows that dF/dz must vanish at B_z, so that B_z is a critical point in the z-plane, though B in the z-plane is not a critical point.

From (2.1),

$$\frac{dF}{dz} = Ue^{-i\alpha} - \frac{Ua^2e^{i\alpha}}{(z - z_0)^2} + \frac{i\Gamma}{2\pi(z - z_0)}.$$

The center of the cylinder $\mathfrak{G}$ is at $z_0 = 0$ and its radius is $a = 1/|K|$. Since at the tail point $z = 1/K$, dF/dz vanishes, it follows that[4]

$$\Gamma = 2\pi Ui\left[\frac{e^{-i\alpha}}{K} - \frac{e^{i\alpha}}{K^*}\right] = 4\pi U \operatorname{imag}\left(\frac{e^{i\alpha}}{K^*}\right), \tag{4.12}$$

which is the value of Γ to be used in (2.1).

5. **Relation to previous papers.** The family $\mathfrak{B}$ of airfoils given in §§3, 4 is somewhat more general than the family $\mathfrak{F}$ advanced in [*PPP*]. The family $\mathfrak{F}$ is obtained from the family $\mathfrak{B}$ by setting $\lambda = 1$. The hyperbolas used in $\mathfrak{F}$ contain the single parameter σ, the varying of

[4] K^* is conjugate of K.

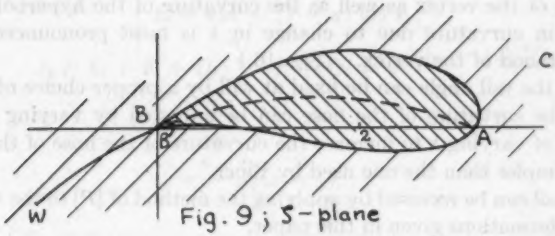

Fig. 9 ; S-plane

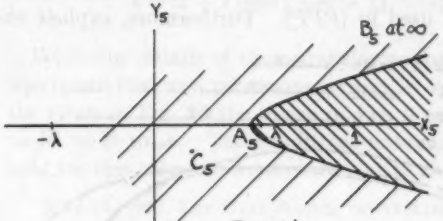

Fig. 10 ; Z_5-plane

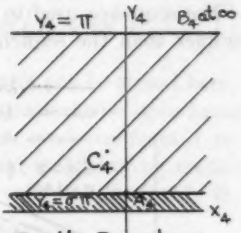

Fig. 11 ; Z_4-plane

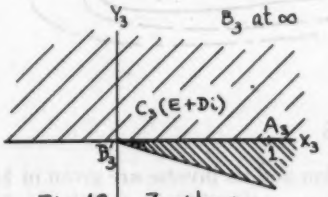

Fig. 12 Z_3-plane

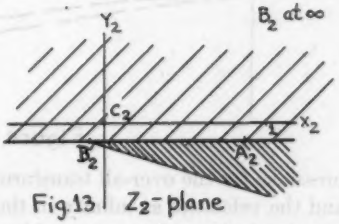

Fig. 13 Z_2-plane

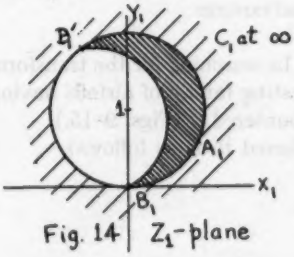

Fig. 14 Z_1-plane

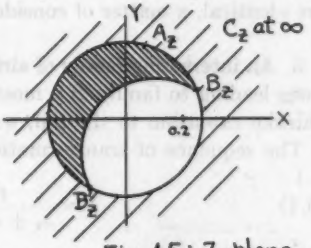

Fig. 15 ; Z-plane

which changes the angle between the asymptotes and hence the tail angle of the profile. In addition to the parameter σ, the family $\mathfrak{B}$ contains the parameter λ. The varying of λ, with σ fixed, changes the

position of the vertex as well as the curvature of the hyperbola. The change in curvature due to change in λ is most pronounced in the neighborhood of the vertex. (Fig. 16.)

Thus the tail angle can be fixed at will by a proper choice of σ, after which the curvature of the nose can be adjusted by varying λ. The method of varying λ to increase the curvature of the nose of the profile seems simpler than the one used by Piper.[5]

The tail can be recessed by applying the method of [*P*] to the sequence of transformations given in this paper.

The sequence used in §§3, 4 to map the profile into a circle seems simpler than the sequence used in [*PPP*]. Furthermore, explicit ex-

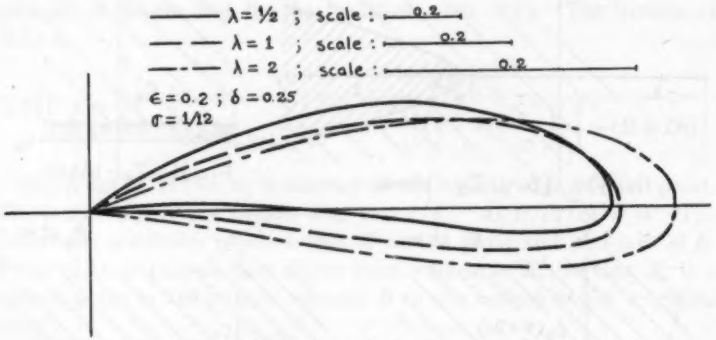

Figure 16

pressions for the over-all transformation and its inverse are given in §4, and the velocities at infinity in the plane of the circle and of the profile are identical, a matter of considerable advantage.

6. **An interesting family of airfoils.** In searching for the transformations leading to family $\mathfrak{B}$ a most interesting family of airfoils having a hairlike extension to the tail, was encountered. (Figs. 9–15.)

The sequence of transformations referred to is as follows:

$$\zeta = \frac{e^{i\beta}}{z_3 + \epsilon + \delta i}, \tag{6.1}$$

$$z_3 = \lambda \cosh z_4, \tag{6.2}$$

[5] Piper, R., "Extensions of the new family of wing profiles." Phil. Mag., Vol. 24, Series 7, (1937), pp. 1114–1126. [P].

$$z_4 = \sigma\pi i + (1 - \sigma)\log z_3, \tag{6.3}$$

$$z_3 = z_2 + E + Di; \qquad E + Di = e^{\frac{1}{1-\sigma}\left[\cosh^{-1}\left(\frac{-\epsilon-\delta i}{\lambda}\right)-\sigma\pi i\right]}, \tag{6.4}$$

$$z_2 = \frac{2D}{z_1}, \qquad D > 0, \tag{6.5}$$

$$z_1 = -i(mz - 1), \tag{6.6}$$

$$\zeta = \frac{e^{i\beta}}{\epsilon + \delta i + \lambda\cosh\left[\pi\sigma i + (1 - \sigma)\log\left(E + Di + \dfrac{2Di}{z - 1}\right)\right]}. \tag{6.7}$$

While the details of these transformations will not be stated here, it was found that as a point moves from A counterclockwise once around the circle in Fig. 15 the image in Fig. 9 moves clockwise from A to B to W to B to A. The conditions encountered in §2 can be made to hold for this case in a manner similar to that exhibited for family $\mathfrak{B}$.

MAY 11, 1940, THE CASE SCHOOL OF APPLIED SCIENCE
CLEVELAND, OHIO.

THE CINEMA INTEGRAPH IN INTERREFLECTION PROBLEMS[1]

By Walter R. Hedeman, Jr.

1. Introduction

The interreflection problem is concerned with the multiple reflection of radiant energy between reflecting surfaces or between parts of the same surface. Yamauti (1) expressed the problem in the form of a finite number of simultaneous linear algebraic equations by dividing the system into elemental parts and considering conditions to be uniform within any one part. Buckley (2) recognized the set of equations so formed as Fredholm's solution of an integral equation of the second kind (5), and stated the problem very concisely as an integral equation. This form of expression has been difficult to utilize. Rigorous solutions have been obtained only for those relatively few cases which are analytically simple (2, 13). Approximate methods of solution have been devised for more complicated problems, but have been applied with difficulty.

The object of this paper is to demonstrate the utility of a computing machine, the *Cinema Integraph*, in the numerical solution of integral equations. A review of the theory of the interreflection problem will be followed by the solution of a particular problem by machine methods. This solution will be compared with the solution obtained for the same problem by Buckley's approximate method.

2. The Integral Equation

Let there be a surface S which reflects according to Lambert's cosine law. The symmetry of the surface is such that the problem is one-dimensional, and the integral equation for the luminosity at any point $P_1(z)$ is (14)

$$L(z) = L_0(z) + \rho(z) \int_s K(z, \zeta) L(\zeta)\, ds(\zeta) \qquad (1)$$

[1] A portion of a thesis entitled "The Numerical Solution of Integral Equations on the Cinema Integraph," accepted by the Massachusetts Institute of Technology in partial fulfillment of the requirements for the degree of Doctor of Science in Electrical Engineering.

where

$L_0(z)$ = luminosity at P_1 previous to interreflections—the "exciting luminosity."
$\rho(z)$ = reflection factor of the surface at the point P_1.
$K(z, \zeta)$ = kernel of the integral equation, which can be interpreted as the illumination at P_1 produced by unit luminosity and unit area at P_2.
$L(\zeta)$ = luminosity of the area ds at P_2.

The particular problem which will be solved is that in which the exciting luminosity varies lengthwise along a right circular cylinder and is constant around any cross-section of the cylinder. Due to the symmetry of the system the total luminosity will vary lengthwise along the cylinder, and Eq. (1) will express the problem completely. If it is further assumed that the reflection factor is constant over the surface of the cylinder and equal to ρ, Eq. (1) becomes

$$L(z) = L_0(z) + \rho \int_s K(z, \zeta) L(\zeta)\, ds(\zeta) \tag{2}$$

The function $K(z, \zeta)\, ds(\zeta)$ in Eq. (2) is the illumination produced at z by a narrow band of area ds and of unit luminosity at $z = \zeta$.

As stated by Buckley (2), the kernel for the circular cylinder is obtained from the expression:

$$K(z - \zeta)\, ds(\zeta) = \frac{1}{2a}\left\{1 - \frac{(z - \zeta)[(z - \zeta)^2 + 6a^2]}{[(z - \zeta)^2 + 4a]^{3/2}}\right\} d\zeta \tag{3}$$

where a = radius of cylinder. This function can be inserted in Eq. (2). The limits of integration are 0 and b, where b is the length of the cylinder parallel to its axis. By the changes of variable:

$$\nu = \frac{z}{a}, \qquad \nu_1 = \frac{\zeta}{a},$$

Eq. (3) becomes

$$K(\nu - \nu_1)\, ds(\nu_1) = \frac{1}{2}\left\{1 - \frac{(\nu - \nu_1)[(\nu - \nu_1)^2 + 6]}{[(\nu - \nu_1)^2 + 4]^{3/2}}\right\} d\nu_1 \tag{4}$$

and the integral equation for the cylinder in these coordinates is

$$L(\nu) = L_0(\nu) + \rho \int_0^{b/a} K(\nu - \nu_1)\, L(\nu_1)\, d\nu_1 \tag{5}$$

where

$$K(\nu - \nu_1) = \frac{1}{2}\left\{1 - \frac{(\nu - \nu_1)[(\nu - \nu_1)^2 + 6]}{[(\nu - \nu_1)^2 + 4]^{3/2}}\right\} \tag{6}$$

In the accepted nomenclature (5), Eq. (5) is known as a linear integral equation of the second kind, Fredholm type.

3. Solution of the Fredholm Equation

It has been stated by Volterra (6) that the solution of Eq. (5) can be written:

$$L(\nu) = L_0(\nu) + \int_0^{b/a} \Gamma(\rho; \nu, \nu_1)\, L_0(\nu_1)\, d\nu_1, \tag{7}$$

where $\Gamma(\rho; \nu, \nu_1)$, known as the *reciprocal kernel* [that is, reciprocal to $\rho K(\nu, \nu_1)$], is defined by the equation

$$\Gamma(\rho; \nu, \nu_1) = K(\nu, \nu_1) + \rho \int_0^{b/a} K(\nu, s)\, \Gamma(\rho; s, \nu_1)\, ds \tag{8}$$

The solution of Eq. (8) is given by Volterra (6), and also by Bôcher (7). Define the series of functions:

$$\left.\begin{aligned} K_1(\nu, \nu_1) &= K(\nu - \nu_1) \\ &\cdots\cdots\cdots\cdots\cdots \\ K_n(\nu, \nu_1) &= \int_0^{b/a} K_1(\nu, s)\, K_{n-1}(s, \nu_1)\, ds \end{aligned}\right\} \tag{9}$$

where $K_n(\nu, \nu_1)$ are known as *iterated functions*. Then it can be shown

$$\Gamma(\rho; \nu, \nu_1) = \sum_{n=1}^{\infty} \rho^n K_n(\nu, \nu_1). \tag{10}$$

If $\Gamma(\rho; \nu, \nu_1)$ is known, it can be substituted in Eq. (7) to give the solution of Eq. (5) for any exciting luminosity $L_0(\nu)$.

It is noted that Eq. (9) is independent of the reflection factor. The only parameter entering the equation is the shape factor b/a; that is, the ratio of the length of the cylinder to the radius of the base. The evaluation of Eq. (9) for all ratios b/a will then conclude the solution of Eq. (5) for all right circular cylinders, and for all exciting luminosities.

4. The Evaluation of Definite Integrals

The solution of Eq. (8) involves the evaluation of a large number of definite integrals, these integrals being stated by Eq. (9). Approximate evaluations of such integrals are based on approximations to the func-

tion $K(\nu, \nu_1)$, as, for instance, Buckley's exponential approximation for the example given (2). It is mentioned in passing that when an exponential approximation to the kernel can be found, the solution is obtainable by Whittaker's (8) method. In many cases, however, such approximations lead to errors of unpredictable magnitude in the final solution of the problem. It is here that machine computation as performed by the Cinema Integraph is of great advantage; that is, the machine can be used equally well for evaluating the accuracy of a given approximate solution and for solving the integral equation. The methods of performing these operations will be demonstrated later in this paper.

The Cinema Integraph, briefly,[2] is a machine that measures a quantity of light which is proportional to the integral being evaluated. The first

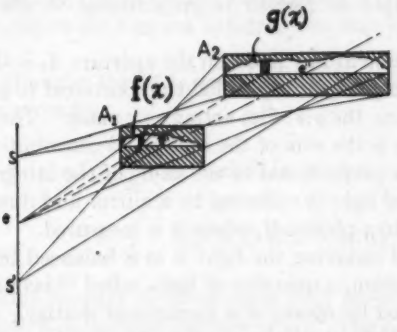

FIG. 1. Principle of operation of the Cinema Integraph

operation which it performs is that of multiplying two functions point-by-point; the second operation is the integration of this product between definite limits. The method of doing this is shown in Fig. 1.

In this figure SS' is a line source of light having substantially uniform radiation per unit length. To simplify the discussion, say that SS' radiates "light," and that the radiation obeys Lambert's Law (4). The apertures A_1 and A_2 are located at e' and e'', respectively, on the optical axis $ee'e''$ so that ee' is equal to $e'e''$. The surfaces of these apertures are coincident with the surfaces of right circular cylinders of which SS' is the axis.

To evaluate the definite integral $\int_a^b f(x)g(x)\,dx$, a strip of 70-mm.

[2] For a more complete discussion of the Cinema Integraph, see Hazen and Brown (9).

motion-picture film is prepared for $f(x)$ with transparent and opaque regions such that the transparent ordinate at any value of x is proportional to $f(x)$, and so that the boundary between transparent and opaque regions represents the function $f(x)$. This film is inserted in the aperture A_1 ; and a similar film, prepared for the function $g(x)$, is inserted in the aperture A_2 .

The films are placed in their respective apertures and "synchronized" so that corresponding values of x lie on straight lines passing through the light source (the abscissa scale for A_2 is twice the abscissa scale for A_1). Consider a single, narrow, vertical, transparent element of the film in the aperture A_1 . This element can be imagined as composed of a vertical chain of small, rectangular holes. Any one of these holes will form an image of the light source SS' on the aperture A_2 . The number of images so formed is proportional to the total ordinate, or $f(x)$.

The illumination at any point on the aperture A_2 is then proportional to $f(x)$. But only a quantity of light proportional to $g(x)$ is allowed to pass this aperture, the $g(x)$ film acting as a mask. The total amount of light passing A_2 is the sum of the individual contributions from narrow elements, and is proportional to the value of the integral stated. This total quantity of light is collected by a mirror and directed through an optical system to a photocell, where it is measured.

The photocell receiving the light is in a balanced bridge circuit and receives, in addition, a quantity of light called "bias light." This bias light is controlled by means of a mechanical shutter. With the bridge in balance, the light that is proportional to the integral is admitted to the photocell. The mechanical shutter on the bias light is then closed to reduce the bias light sufficiently to obtain a balance of the bridge. The amount that the shutter closes is directly proportional to the value of the integral. Balance is restored automatically by means of a servomotor which drives the mechanical shutter through a torque amplifier.

Various mechanisms are provided on the Cinema Integraph to permit simplification of procedure for special types of mathematical functions. Functions that are of especial interest to this paper are so-called "shifting" functions—a type encountered in the superposition theorem for linear networks.[3] Suppose the integral to be evaluated is

[3] It is noted that many functions $f(t, T)$ can be transformed into shifting functions in order to take advantage of the continuous evaluation method. The means of finding whether such a transformation is possible, and also of finding the necessary changes of variables, is given by C. E. Shannon in Appendix II of Reference 11.

$\int_a^b f(t - T)g(T)\,dT$. This integral will result in a functional evaluation in terms of t, which is a parameter in this case. A film is prepared for $g(T)$ and another for $f(T)$.

When t is zero, it is desired to evaluate $h(0) = \int_a^b f(-T)g(T)\,dT$. The function $f(-T)$ is simply the function $f(T)$ turned end-for-end. To add t, in obtaining $f(t - T)$, involves shifting $f(-T)$ to the right in the aperture by the amount t. The film is driven continuously through the aperture by a motor drive, and balance is continuously maintained by the servo-motor. The shutter is geared to an output stylus which writes on a platen geared to the film displacement. As the integral is continuously evaluated as a function of t, the output stylus draws a graph of the output function. If calibrated graph paper is used on the platen, it is possible to adjust the Cinema Integraph so that this graph will be direct-reading.

Suppose, for example, that it is desired to evaluate $K_2(\nu, \nu_1)$, Eq. (9). This function is given by

$$\left.\begin{aligned} K_2(\nu, \nu_1) &= \int_0^{b/a} K(\nu, s)\, K_1(s, \nu_1)\, ds \\ &= \int_0^{b/a} K(\nu - s)\, K(s - \nu_1)\, ds. \end{aligned}\right\} \tag{11}$$

Since Eq. (7) later requires an integration with respect to ν_1, it is desirable to evaluate $K_2(\nu, \nu_1)$ as a function of ν_1. The functions involved are shifting functions, so a continuous evaluation in this parameter is possible.

The method used to evaluate $K_2(\nu, \nu_1)$ was to prepare two films for $K(s)$. One of these was made for the aperture A_1, and the second for the aperture A_2. One of these films was shifted by the quantity ν, and remained fixed. The other film was shifted continuously by the quantity ν_1, where $0 \leqq \nu_1 \leqq b/a$. As this second film was shifted continuously, the integral of Eq. (11) was evaluated continuously as a function of the parameter ν_1 and was recorded as a graph on the output platen. $K_2(\nu, \nu_1)$ was evaluated for a sufficiently large number of different values of ν so that interpolation between functions was possible.[4]

[4] Bôcher also notes some properties of iterated functions corresponding to certain properties of the kernel. Such mathematical laws often provide material aid in solving integral equations and in evaluating the accuracy of solutions obtained by machine methods.

The evaluation of $K_3(\nu, \nu_1)$ proceeds in much the same way, except that several frames of film must be prepared for $K_2(\nu, s)$, as this function is not a shifting function. Therefore a frame of film was prepared for each value of ν for which $K_2(\nu, s)$ was evaluated. These frames were placed successively in the aperture of the Cinema Integraph and $K_3(\nu, \nu_1)$ evaluated; that is, for a particular value of ν, $K_2(\nu, s)$ was placed in one aperture and $K(s)$ in the other aperture. If $K(s)$ in the one aperture is shifted continuously by the quantity ν_1, then $K_3(\nu, \nu_1)$ is evaluated continuously in the parameter ν_1 for the particular value of ν corresponding to the film for $K_2(\nu, s)$. This process was repeated for that number of different values of ν for which $K_2(\nu, \nu_1)$ was evaluated.

The relation between iterated functions is illustrated graphically in the above process. In this process the result of the evaluation of a parametric product integral is placed under the integral sign, and this result is in turn reinserted under the integral sign.

5. Evaluation of Errors

In any computational procedure involving the use of machine aids, sources of error must be carefully eliminated. With respect to the Cinema Integraph, certain general checks have been found to be sufficient. The accuracy of a reciprocal kernel can be estimated. Say, for example, that $\Gamma(\rho; \nu, \nu_1)$ has been evaluated and that the function obtained by evaluating the integral of Eq. (7) is $L_1(\nu)$, different from $L(\nu)$. This function $L_1(\nu)$ will satisfy an integral equation, and this equation is

$$L_1(\nu) = L_{01}(\nu) + \rho \int_0^{b/a} K(\nu - \nu_1)\, L_1(\nu_1)\, d\nu_1 . \tag{12}$$

Eq. (12) is then a definition of the function $L_{01}(\nu)$.

Subtracting Eq. (12) from Eq. (5), and letting

$$L_R(\nu) = L(\nu) - L_1(\nu) \tag{13a}$$

$$L_{R0}(\nu) = L_0(\nu) - L_{01}(\nu) \tag{13b}$$

we obtain

$$L_R(\nu) = L_{R0}(\nu) + \rho \int_0^{b/a} K(\nu - \nu_1)\, L_R(\nu_1)\, d\nu_1 . \tag{14}$$

From Eq. (7),

$$L_R(\nu) = L_{R0}(\nu) + \int_0^{b/a} \Gamma(\rho; \nu, \nu_1)\, L_{R0}(\nu_1)\, d\nu_1 . \tag{15}$$

The percent accuracy of $L_1(\nu)$ is then:

$$\eta(\nu) = 100 \frac{L_1(\nu)}{L_1(\nu) + L_R(\nu)}. \tag{16}$$

If $L_R(\nu)$ is small, however, the integral of Eq. (14) can be neglected, and

$$\eta(\nu) \cong 100 \frac{L_1(\nu)}{L_1(\nu) + L_{R0}(\nu)}. \tag{17}$$

From Eqs. (12) and (13b),

$$L_1(\nu) + L_{R0}(\nu) = L_0(\nu) + \rho \int_0^{b/a} K(\nu - \nu_1)\, L_1(\nu_1)\, d\nu_1. \tag{18}$$

Introduce the notation:

$$L_2(\nu) = L_1(\nu) + L_{R0}(\nu). \tag{19}$$

Thus,

$$\eta(\nu) = 100 \frac{L_1(\nu)}{L_2(\nu)}. \tag{20}$$

It is interesting to note the similarity between this process of evaluating errors and the process of solving an integral equation by successive approximations (5). In the method of successive approximations, the function defined here as $L_2(\nu)$ is exactly the second approximation to $L(\nu)$ if $L_1(\nu)$ is the first approximation to $L(\nu)$.

Consider the particular problem where $L_0(\nu)$ is exactly unity. For this problem, by Eq. (7),

$$L_1(\nu) = 1 + \int_0^{b/a} \Gamma(\rho; \nu, \nu_1)\, d\nu_1, \tag{21}$$

and, from Eq. (18),

$$L_2(\nu) = 1 + \rho \int_0^{b/a} K(\nu - \nu_1)\, L_1(\nu_1)\, d\nu_1. \tag{22}$$

The difference between $L_1(\nu)$ and $L_2(\nu)$ is a measure of the accuracy of $L_1(\nu)$. The integral of Eq. (21), evaluated in finding $L_1(\nu)$, is the area under $\Gamma(\rho; \nu, \nu_1)$ for a particular value of ν. The indicated error is then an error in the area of $\Gamma(\rho; \nu, \nu_1)$. Since the function is essentially integrated when applied in the solution of a particular problem, this is accepted as a sufficient measure of its accuracy.

This procedure can be utilized also by employing more accurate com-

putational methods than the Cinema Integraph. That is, an approximate solution can be obtained rapidly by means of the Cinema Integraph, the error evaluated by means of a more accurate medium such as the Differential Analyzer (10), and the error equation, Eq. (15), solved again on the Cinema Integraph (11).

6. Solution of a Particular Problem

The series of functions $K_n(\nu, \nu_1)$ was evaluated for the circular cylinder with shape factor $b/a = 2$. The series is convergent at every point, and it was found that terms beyond the eighth were of the order of

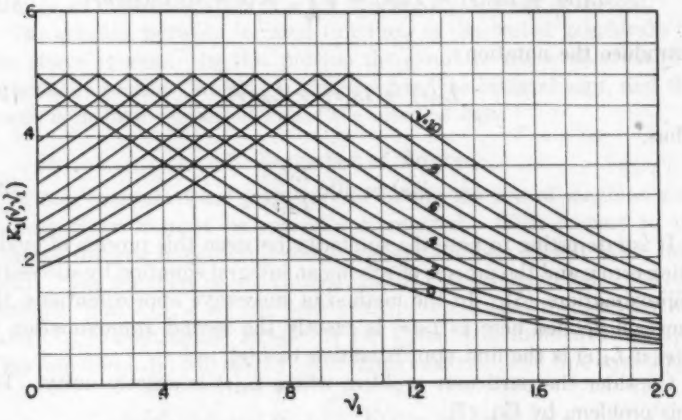

FIG. 2a. Iterated kernels for the luminosity distribution in a right circular cylinder

magnitude of the probable error and could be neglected. Figures 2a to 2h show the functions $K_1(\nu, \nu_1)$ to $K_8(\nu, \nu_1)$, respectively. Final results for the luminosity distribution are given in Fig. 3 and Table I.

Table I gives $L(\nu)$ from Eq. (12), and the first correction to this solution from Eq. (13). The corrected solution was computed manually in this case by means of an average-ordinate method, but usually this computation is also performed on the Cinema Integraph. Table I shows in addition the solution obtained by Buckley's method, using a single-term exponential approximation to $K(\nu, \nu_1)$.

It is noted from Table I that the estimated error of the reciprocal-kernel solution is everywhere less than one percent, while Buckley's

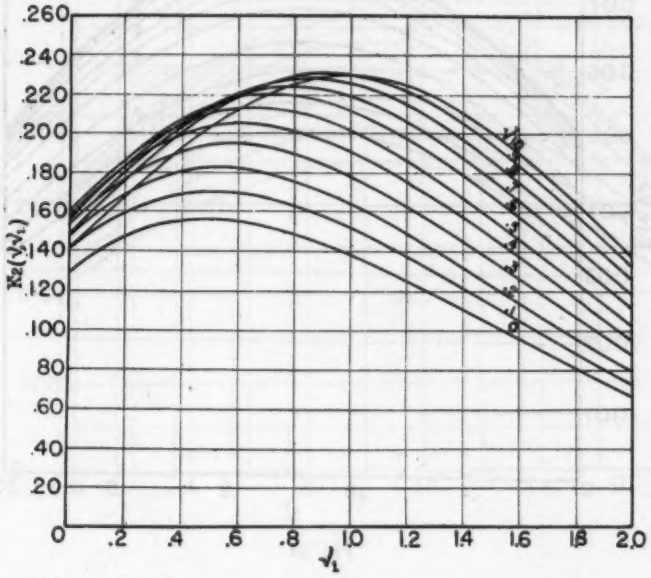
.260
.240
.220
.200
.180
.160
.140
.120
.100
.80
.60
.40
.20
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
$K_2(\gamma,\gamma_1)$
γ_1
γ_{10}
8
7
6
5
4
3
2
1
0

FIG. 2b

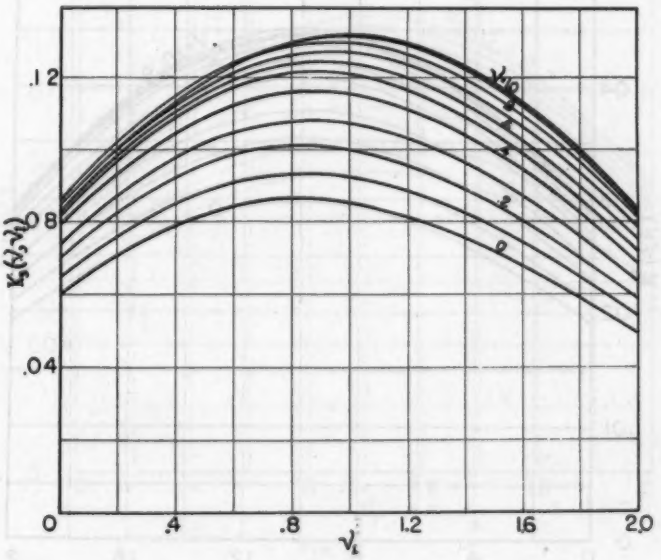
.12
.08
.04
0
.4
.8
1.2
1.6
2.0
$K_3(\gamma,\gamma_1)$
γ_1
γ_{10}
8
7
4
2
0

FIG. 2c

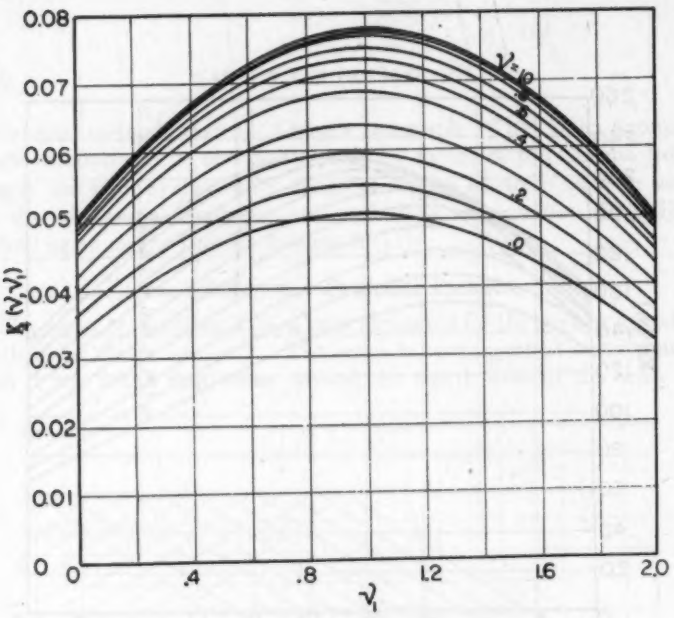
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
.4
.8
1.2
1.6
2.0
ν_1
$K_4(\nu,\nu_1)$
$\nu=1.0$
.8
.6
.4
.2
.0

FIG. 2d

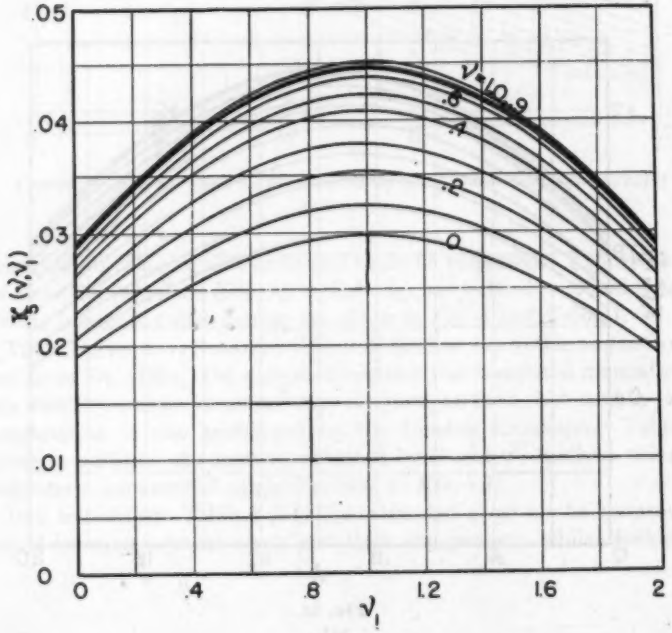
.05
.04
.03
.02
.01
0
0
.4
.8
1.2
1.6
2
ν_1
$K_5(\nu,\nu_1)$
$\nu=1.0$
.9
.8
.4
.2
0

FIG. 2e

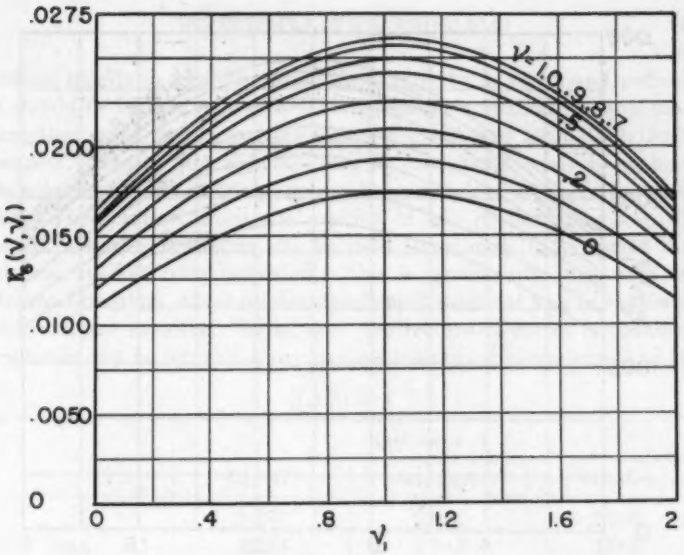
.0275
.0200
.0150
.0100
.0050
0
$K_0(\nu,\nu_1)$
ν= 1.0, .9, .8, .7
.5
.2
0
0
.4
.8
1.2
1.6
2
ν_1

FIG. 2f

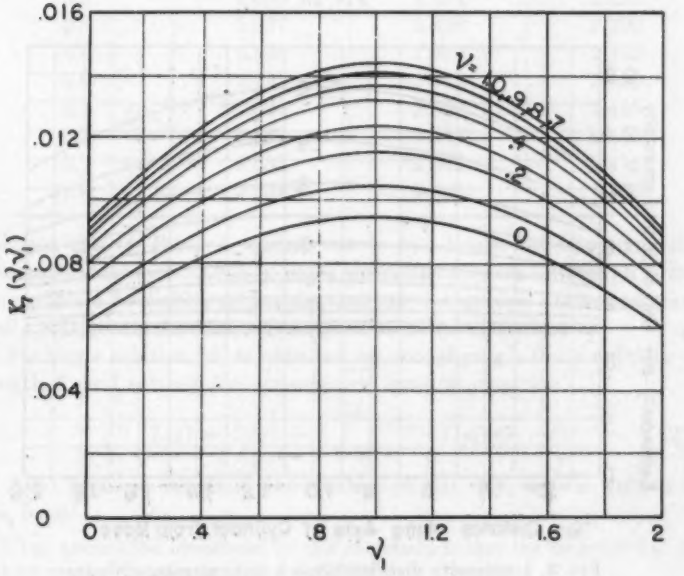
.016
.012
.008
.004
0
$K_1(\nu,\nu_1)$
ν= 1.0, .9, .8, .7
.5
.2
0
0
.4
.8
1.2
1.6
2
ν_1

FIG. 2g

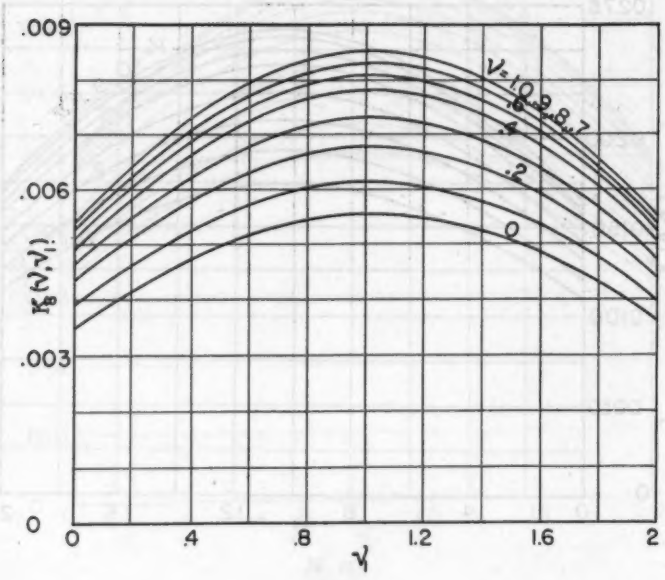

FIG. 2h

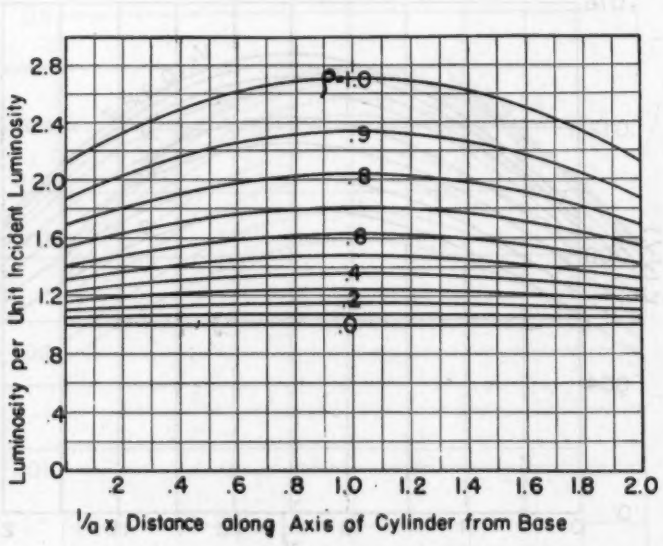

FIG. 3. Luminosity distribution in a right circular cylinder
L_e = const. $b = 2a$

method results in a solution which is in error by as much as 4 percent. It should be further pointed out that Buckley's method requires more analytical work as the exciting luminosity becomes more analytically complex. The reciprocal kernel may be used immediately to evaluate the solution for any exciting luminosity by means of an explicit integration. Furthermore, when the solution of any problem has been obtained, a simple procedure can be used to estimate the error of that solution, as has been indicated. This is particularly easy with the Cinema Integraph, where solutions and check solutions may be evaluated with extreme rapidity. Thus, once the reciprocal kernel is obtained, a solution can be found for any particular exciting luminosity; and the

TABLE I

ν	$L(\nu)$ for $\rho = 1$		
	Eq. (12)	1st. Corr.	Buckley
0	2.036	2.044	2.000
0.1	2.134	2.147	2.095
0.2	2.223	2.240	2.180
0.3	2.305	2.324	2.255
0.4	2.377	2.398	2.320
0.5	2.440	2.462	2.375
0.6	2.493	2.514	2.420
0.7	2.532	2.555	2.455
0.8	2.563	2.585	2.480
0.9	2.583	2.602	2.495
1.0	2.584	2.608	2.500

solution can be checked, within one to two hours, by employing the Cinema Integraph. Even in cases where the Cinema Integraph is not immediately available, an average-ordinate integration performed manually will give a solution within a reasonably short time.

Buckley's solution (2) is obtained by considering a finite cylinder of length 2, and solving the approximate integral equation[5]

$$L(\nu) = L_0(\nu) + \frac{\rho}{2}\int_{-1}^{1} e^{-|\nu-\nu_1|} L(\nu_1)\, d\nu_1 . \tag{23}$$

If $L_0(\nu) = L_0 =$ constant, the solution of Eq. (23) can be written in the form:

[5] The assumption introduced by this expression is that the kernel of Eq. (6) can be expressed approximately by $e^{-|\nu-\nu_1|}$.

$$L(\nu) = a_0 + a_1 \cosh b\nu. \tag{24}$$

Substitution of Eq. (24) in Eq. (23) gives

$$b = \sqrt{1 - \rho} \tag{25a}$$

$$a_0 = \frac{L_0}{1 - \rho} \tag{25b}$$

$$a_1 = -\frac{a_0 \rho}{\cosh bl + b \sinh bl} \tag{25c}$$

The solution is thus:

$$L(\nu) = \frac{L_0}{1 - \rho}\left\{1 - \frac{\rho \cosh \sqrt{1 - \rho}\,\nu}{\cosh \sqrt{1 - \rho}\,l + \sqrt{1 - \rho} \sinh \sqrt{1 - \rho}\,l}\right\}. \tag{26}$$

When $\rho = 1$, the quotient term in brackets becomes indeterminate. By the use of l'Hospital's Rule (12) it can be shown that, for $\rho = 1$,

$$L(\nu) = \frac{L_0}{2}[(l + 1)^2 + 1 - \nu^2] \tag{27}$$

In Eq. (27) the symbol "l" is identical with $b/2a$, or one-half the "shape-factor," as it is called in this paper. For the particular case where $b/a = 2$, Eq. (27) becomes

$$L(\nu) = \frac{L_0}{2}[5 - \nu^2], \qquad (-1 \leqq \nu \leqq 1) \tag{27a}$$

and when $L_0 =$ unity, this becomes

$$L(\nu) = \tfrac{1}{2}[5 - \nu^2], \qquad (-1 \leqq \nu \leqq 1) \tag{27b}$$

Eq. (27b) is that from which the comparison solution given in Table I is derived, and is listed as "Buckley's solution."

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ON THE MOTION OF AN ELECTRIC PARTICLE

By I. Opatowski

The bases for a theoretical study of the motion of an electric particle in a magnetic dipole field were given by C. Störmer ([1]): he reduced the problem to that of a motion of the particle in a plane and to a rotation of that plane about an axis. He proved also the existence of forbidden regions i.e. regions where no motion is possible. C. Graef and S. Kusaka gave explicit formulas for certain plane motions in a dipole field ( ). C. Störmer still in 1912 pointed out the importance of the study of the motion in *general* electric and magnetic fields ([2]). The recent investigations of cosmic rays caused new interest in this problem ([4]) ([5]). In the present paper the particle is assumed to move in electric and magnetic fields of a very general form which include the most important fields of mathematical physics (fields symmetric about an axis, plane fields, etc.); forbidden regions are determined and cases are indicated where the equation of motion can be simplified or explicitly integrated.

1. The Integral of Energy. The equation of motion of an electric particle in electric and magnetic fields of intensity $\mathbf{E}_1$ and $\mathbf{H}_1$ respectively is ([6]) ([7]):

$$(1)\quad d(m\mathbf{v})/dt = \mathbf{E} + \mathbf{v} \times \mathbf{H}, \quad \text{where} \quad \mathbf{E} = e\mathbf{E}_1\,, \quad \mathbf{H} = ec^{-1}\mathbf{H}_1\,,$$

$\mathbf{v}$ is the velocity of the particle, c the velocity of the light, m the mass (function of v), e the electric charge and $\times$ the symbol of the right-handed vector product ([7]). We shall use the right-handed system of Cartesian coordinates x, y, z and ∇ as the symbol of the gradient. Putting $\mathbf{E} = -\nabla V$ the scalar product of (1) by $\mathbf{v}$ gives the *integral of energy*:

$$(2)\qquad mv^2 - \tfrac{1}{2}\int m\cdot dv^2 + V = K,$$

where K is a constant. Eq. (2) shows that the variation of the magnitude of the velocity depends only on the electric field and not on the magnetic. The motion of the particle on an equipotential surface of the electric field is possible only with a constant velocity.

From Eq. (2) we have for *constant mass*:

$$(3)\qquad v^2 = 2(K - V)/m, \quad \text{and}$$

$$(4)\qquad v^2 = c^2[1 - c^4m_0^2(K - V)^{-2}]$$

for *relativistic mass* if m_0 is the mass at rest. It is seen therefore that the motion is possible only where $V < K$ if the mass is constant or where $m_0c^2 < K - V$ if the mass is relativistic. If the trajectory of the particle passes through infinity and all electric charges are in a finite region, then $V_\infty = 0$ and $K > 0$ or $> m_0c^2$ respectively.

2. The Integral of Momentum. In this and in the following Secs. we consider fields defined by the existence of two constant vectors $\mathbf{a}$ and $\mathbf{b}$, a scalar function W and a scalar constant C such that:

$$(5) \qquad \nabla W = \mathbf{u} \times \mathbf{H}, \quad -C = \mathbf{u} \cdot \mathbf{E}, \quad \text{where} \quad \mathbf{u} = \mathbf{a} + \mathbf{b} \times \mathbf{r},$$

$\mathbf{r}$ being the radius vector. Since $\mathbf{v} = d\mathbf{r}/dt$, we have: $\mathbf{v} \cdot d\mathbf{u}/dt = 0$, so that $\mathbf{u} \cdot d(m\mathbf{v})/dt = d(m\mathbf{v} \cdot \mathbf{u})/dt$ and the scalar product of (1) by $\mathbf{u}$ gives: $d(m\mathbf{v} \cdot \mathbf{u})/dt = \mathbf{E} \cdot \mathbf{u} + \mathbf{v} \times \mathbf{H} \cdot \mathbf{u}$ or by (5): $d(m\mathbf{v} \cdot \mathbf{u})/dt = -C - \mathbf{v} \cdot \nabla W$. But $\mathbf{v} \cdot \nabla W = dW/dt$, hence by integration:

$$(6) \qquad m\mathbf{v} \cdot \mathbf{u} + W = L - Ct, \qquad \text{where } L \text{ is a constant.}$$

Eq. (6) gives the *component of the momentum of the particle along the vector* $\mathbf{u}$. Putting in (1) $\mathbf{H} = \nabla W \times \mathbf{u}/u^2$ (which is a consequence of the first Eq. 5), eliminating $\mathbf{v} \cdot \mathbf{u}$ by means of (6) and taking into account that $\mathbf{v} \cdot \nabla W = dW/dt$, we get:

$$(7) \quad d(m\mathbf{v})/dt = \mathbf{E} - u^{-2}[(2m)^{-1}\nabla(L - Ct - W)^2 + (dW/dt)\mathbf{u}].$$

3. Forbidden Regions. We shall see in Sec. 4 that the most important fields correspond to $C = 0$. The absolute value of the cosine of the angle between $\mathbf{v}$ and $\mathbf{u}$ is then by (6): $|(L - W)/(mvu)|$ which must be $\leqq 1$. Eliminating here v by means of (3) and (4) we obtain the forbidden region:

$$(8) \qquad |L - W| > +[2m(K - V)]^{1/2}\,|u| \qquad \text{for constant mass,}$$

$$(9) \quad |L - W| > +[(K - V)^2 - m_0^2c^4]^{1/2}\,|u|\cdot c^{-1} \quad \text{for relativistic mass.}$$

A particle having its energy and the component of its momentum determined by the constants K and L can not reach any of the points satisfying (9) *and* (8). The well known formula which Störmer deduced for the study of aurora borealis and M. S. Vallarta applied in his theory of cosmic rays is a particular case of (8).

4. Fields Satisfying the Condition (5). We shall consider potential fields and put: $\mathbf{H} = -\nabla U$, $\mathbf{E} = -\nabla V$, where U and V are harmonics.

Then it follows from (5): $\nabla W \cdot \nabla U = 0$, i.e. $W = \text{const.}$ represents a family of surfaces orthogonal to the equipotential surfaces of the magnetic field.

Let z, ρ, φ be cylindrical coordinates and $\nu = \epsilon z + n\varphi$, where n is a constant, $\epsilon = 0$ if $n = 1$, $\epsilon = 1$ if $n \neq 1$. ν, ρ, φ are helical coordinates. Assume that:

$$U = f(\rho, \nu) + p(\epsilon\varphi + nz); \qquad V = g(\rho, \nu) + q(\epsilon\varphi + nz), \tag{10}$$

where f and g are functions of ρ and ν only, and p, q are two constants. If $\epsilon = 0$, ν becomes φ and f, g are plane harmonics. If $n = 0$, ν becomes z and f, g are harmonics symmetric about an axis. (10) represents therefore a very important class of fields. It is not difficult to prove that U and V of (10) satisfy the condition (5) with

$$\begin{gathered} \mathbf{a} = -n\nabla z, \qquad \mathbf{b} = \epsilon\nabla z, \qquad -C = q(n^2 - \epsilon), \\ W = -w - pn\epsilon\,(\log\rho + \rho^2/2), \end{gathered} \tag{11}$$

w being a function of ρ, ν defined by:

$$w_\rho = \rho^{-1}(n^2 + \epsilon\rho^2)f_\nu, \qquad w_\nu = -\rho f_\rho \tag{12}$$

or by the equivalent vector equation:

$$\nabla w = (-n\nabla z + \epsilon\nabla z \times \mathbf{r}) \times \nabla f, \tag{13}$$

which may be written also as:

$$\nabla w = (-n\nabla z + \epsilon\rho^2\nabla\varphi) \times \nabla f = \rho(\nabla\nu \times \nabla\rho) \times \nabla f. \tag{13'}$$

The subscripts in (12) mean derivatives. The existence of w is a consequence of the fact that f satisfies the Laplace equation ([3]). A method of calculating w by series expansions has been given by the writer ([8]) ([12]). w reduces to the ordinary plane (or symmetric) stream function if $\epsilon = 0$ (or $n = 0$). Since $\mathbf{v} = \dot\rho\nabla\rho + \rho^2\dot\varphi\nabla\varphi + \dot z\nabla z$, the integral of momentum (6) is by (11) and (13'):

$$m(\epsilon\rho^2\dot\varphi - n\dot z) = w + pn\epsilon\,(\log\rho + \rho^2/2) + L - q(\epsilon - n^2)t. \tag{14}$$

5. The Case of Constant Mass. If $\epsilon = 0$, i.e. if f and g are *plane harmonics*, $\mathbf{u}$ becomes $-\nabla z$ and the projection of (7) on the plane (x, y) perpendicular to $\mathbf{u} = -\nabla z$ is:

$$m(\ddot x\nabla x + \ddot y\nabla y) = -\nabla[g + (2m)^{-1}(L + qt + w)^2]. \tag{15}$$

If $n = 0$, i.e. if f and g are *harmonics symmetric about an axis*, $\mathbf{u}$ becomes $\rho^2\nabla\varphi$. The projection of (7) on the plane (ρ, z) perpendicular to $\mathbf{u}$ and the elimination of $\dot\varphi$ by means of (14) gives then:

$$m(\ddot\rho\nabla\rho + \ddot z\nabla z) = -\nabla[g + (2m\rho^2)^{-1}(L - qt + w)^2]. \tag{16}$$

In both cases, (15) and (16) reduce the motion to a plane motion in the potential field $[\cdots]$ and to the motion of the plane itself according to (14), which is a translation parallel to z-axis in the first case ($\epsilon = 0$) and a rotation about the same axis in the second case ($n = 0$).

6. The Case of Relativistic Mass. Since

$$d(m\mathbf{v})/dt = [D_t(m\dot\rho) - m\rho\dot\varphi^2]\nabla\rho + \rho[\rho D_t(m\dot\varphi) + 2m\dot\rho\dot\varphi]\nabla\varphi + D_t(m\dot z)\nabla z,$$

where $D_t = d/dt$, we get from (7) and (10):

$$\begin{cases} d(m\dot\rho)/dt = m\rho\dot\varphi^2 - g_\rho + m^{-1}(n^2 + \epsilon\rho^2)^{-1}(L - Ct - W)W_\rho, \\ \rho^2\, d(m\dot\nu)/dt = -2mn\rho\dot\rho\dot\varphi - (n^2 + \epsilon\rho^2)g_\nu \\ \qquad\qquad - qn\epsilon(1 + \rho^2) + m^{-1}(L - Ct - W)W_\nu, \end{cases} \tag{17}$$

where:

$$m = m_0c(c^2 - v^2)^{-1/2} = m_0c[c^2 - \dot\rho^2 - \rho^2\dot\varphi^2 - \dot z^2]^{-1/2}. \tag{18}$$

If $\epsilon = 0$, ν is $= \varphi$ and $\dot z$ can be eliminated in m by means of (14) which gives:

$$m^2 = [m_0^2c^2 + (w + L + qt)^2]/(c^2 - \dot\rho^2 - \rho^2\dot\varphi^2). \tag{19}$$

If $\epsilon = 1$, z is $= \nu - n\varphi$ and (14), (18) give then:

$$\begin{aligned} &m_0^2c^2[(n^2 + \rho^2)\dot\varphi - n\dot\nu]^2 \\ &\qquad = (L - Ct - W)^2[c^2 - \dot\rho^2 - \rho^2\dot\varphi^2 - (\dot\nu - n\dot\varphi)^2], \end{aligned} \tag{20}$$

which enables one to eliminate $\dot\varphi$ in m and in (17). It is seen therefore that in the equations of motion the 3rd variable ($\dot z$ or $\dot\varphi$) can always be eliminated.

7. Explicit Equations of a Motion. Assume that ϵ or n is $= 0$ and $V(-\nu) = V(\nu)$, $U(-\nu) = -U(\nu)$. Then by (10), (12) and (11) $g_\nu(\rho, 0) = 0$; $W_\nu(\rho, 0) = 0$ and the 2nd Eq. (17) is satisfied by $\nu = 0$, whereas the 1st Eq. (17) becomes an ordinary differential equation for $\rho(t)$. (If the mass is relativistic eliminations indicated in Sec. 6 must

be carried out.) $\varphi(t)$ or $z(t)$ can be obtained from (14) after the evaluation of $\rho(t)$. If besides the previous conditions q is $= 0$ we get explicitly from (3) or (4), (10), (14) and (19) or (20):

$$t = \int N \cdot R(\rho) \cdot d\rho; \qquad \epsilon\varphi + nz = \int (\epsilon\rho^{-2} - n)[w(\rho, 0) + L] \cdot R(\rho) \cdot d\rho,$$

where for *constant mass*:

$$N = m; \quad R(\rho) = \{2m[K - g(\rho, 0)] - (n + \epsilon\rho^{-2})[w(\rho, 0) + L]^2\}^{-1/2},$$

and for *relativistic mass*:

$$N = [K - g(\rho, 0)]/c^2;$$

$$R(\rho) = \{c^{-2}[K - g(\rho, 0)]^2 - m_0^2 c^2 - (n + \epsilon\rho^{-2})[w(\rho, 0) + L]^2\}^{-1/2}.$$

8. Application of the Lagrange Function. The Lagrange function corresponding to Eq. (1) is ([9]) ([10]) ([11]):

$$\lambda = \frac{1}{2}\int m \cdot dv^2 - V - \mathbf{A} \cdot \mathbf{v}, \tag{21}$$

where $-\mathbf{A}$ is the vector potential of the magnetic field i.e. rot $\mathbf{A} = \nabla U$. If functions ω_i and Ω_i are known such that $\nabla U = \sum_i \nabla\omega_i \times \nabla\Omega_i$, then $\mathbf{A} = \sum_i \omega_i \nabla\Omega_i$. Therefore by $\nu = \epsilon z + n\varphi$ and by (12) we have:

$$-\nabla U = \nabla W \times \mathbf{u}/u^2 = \nabla W \times \nabla\alpha - \epsilon n(n^2 + \rho^2)^{-1} W_\rho \nabla\rho \times \nabla\nu$$
$$= \nabla W \times \nabla\alpha - \nabla\beta \times \nabla\rho,$$

where

$$\alpha = \epsilon\varphi + \epsilon z - z; \quad \beta = \epsilon n\rho^{-1}[f + np(1 + \rho^2)(n^2 + \rho^2)^{-1}\nu],$$

so that

$$-\mathbf{A} = W\nabla\alpha - \beta\nabla\rho \tag{22}$$

and from (21) by $\mathbf{v} = \dot{\rho}\nabla\rho + \rho^2\dot{\varphi}\nabla\varphi + \dot{z}\nabla z$:

$$\lambda = \frac{1}{2}\int m \cdot dv^2 - V + W\dot{\alpha} - \beta\dot{\rho} \tag{23}$$

Since ϵ is $= 0$ or $= 1$, the substitution of (23) into the equation: $d(\partial\lambda/\partial\dot{\alpha})/dt = \partial\lambda/\partial\alpha$ gives at once (14) by (10) and (11).

From (22) we have:

$$W = (\epsilon - 1 - \epsilon\rho^2)\mathbf{A} \cdot \nabla\alpha \tag{24}$$

which determines W in terms of the vector potential of the magnetic field.

In particular cases some of the previous results concerning the trajectories of motion and the momentum integral have been obtained by different authors ([2]) ([3]) ([13]) ([14]). Of particular interest is here the paper of T. Boggio ([13]).

9. An Example: Motion of an Electrical Particle in a Field due to a Magnetic Dipole and to a Ring of Electricity. Assume the dipole in the origin of the system of coordinates and its magnetic moment M parallel to z-axis. Then the potential is $U_1 = M_1 r^{-2} \cos\theta$, where $M_1 = Me/c$ and θ is the angle between the z-axis and $\mathbf{r}$. The stream function corresponding to U_1 is:

$$w_1 = M_1 r^{-1} \sin^2\theta. \tag{25}$$

Assume that besides the dipole, the magnetic field of an electric current of intensity I is acting on the electric particle. Let $x = y = 0$, $z = \zeta$ be the Cartesian coordinates of the geometric center of the current of radius R and $z = \zeta$ the equation of its plane. The vector potential is ([15]):

$$\mathbf{A}_2 = N_2(\sqrt{\rho}/k)[(1 - \tfrac{1}{2}k^2)F(k) - E(k)]\cdot\nabla\varphi, \text{ where } N_2 = -4I\sqrt{R}\cdot e/c$$

and F, E are Legendre's complete elliptic integrals of modulus k defined by: $k^2 = 4R\rho[(z-\zeta)^2 + (\rho + R)^2]^{-1}$. By (24) and (11) the stream function corresponding to $\mathbf{A}_2$ is:

$$w_2(\rho, z) = N_2(\sqrt{\rho}/k)[(1 - \tfrac{1}{2}k^2)F(k) - E(k)]. \tag{26}$$

If an electrostatic charge of linear density δ is uniformly distributed on a circle of radius R_1 in the plane $z = \zeta_1$, where $x = y = 0$, $z = \zeta_1$ are the coordinates of the center of the circle, then the electrostatic potential acting on the particle is:

$$g = 2e\delta k_1(R_1/\rho)^{1/2}F(k_1), \text{ where } k_1^2 = 4R_1\rho[(z-\zeta_1)^2 + (\rho + R_1)^2]^{-1}. \tag{27}$$

Putting in the formulas of Sec. 3 and 4 $n = 0$, $\epsilon = 1$, $\nu = z$ and $w = w_1 + w_2$, where w_1 and w_2 are given by (25) and (26) and g by (27) we obtain *a forbidden region for the motion of an electric particle in the field due to a simultaneous action of a magnetic dipole, a circular electric current of constant intensity and an electrostatic charge distributed uniformly over a circle.* The equations of motion in the meridian plane can be deduced in the same way from Sec. 5 and 6. If $\zeta_1 = \zeta = 0$, we have by (27):

$g(-z, \rho) = g(z, \rho)$ and by (25), (26): $w_i(-z, \rho) = w_i(z, \rho)$; $(i = 1, 2)$, so that motions in the plane $z = 0$ exist and their explicit equations are given by Sec. 7. The complete determination of the forbidden regions for the fields here considered is of interest for the theory of cosmic rays and has been suggested to the writer by Professor M. S. Vallarta.

UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINN.

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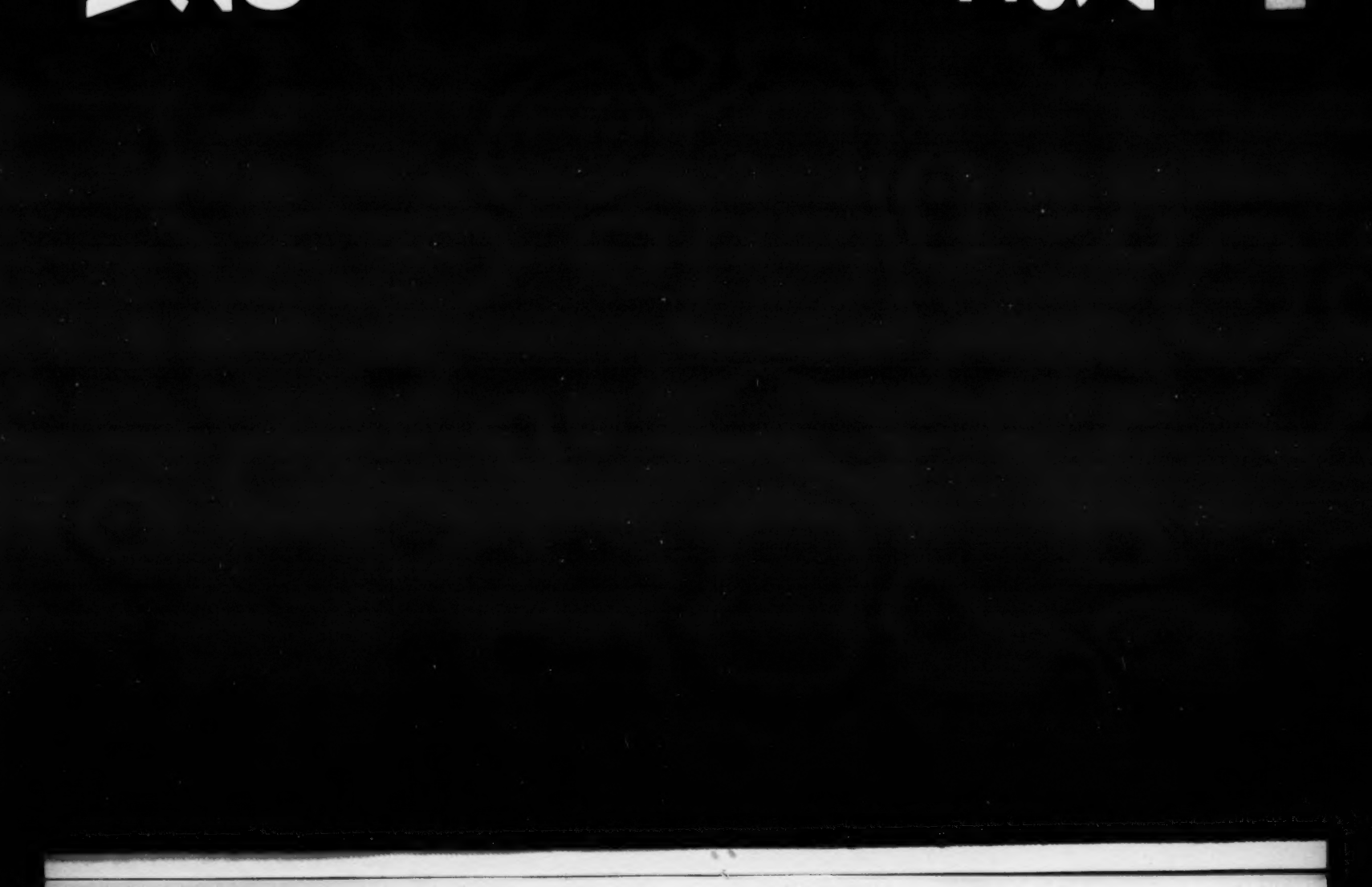
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